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大中华文库

汉英对照

四元玉鉴

JADE MIRROR OF THE FOUR UNKNOWNS

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Chinese-English

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Jade Mirror of the Four Unknowns

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总　　序

杨牧之

《大中华文库》终于出版了。我们为之高兴，为之鼓舞，但也倍感压力。

当此之际，我们愿将郁积在我们心底的话，向读者倾诉。

一

中华民族有着悠久的历史和灿烂的文化，系统、准确地将中华民族的文化经典翻译成外文，编辑出版，介绍给全世界，是几代中国人的愿望。早在几十年前，西方一位学者翻译《红楼梦》，书名译成《一个红楼上的梦》，将林黛玉译为“黑色的玉”。我们一方面对外国学者将中国的名著介绍到世界上去表示由衷的感谢，一方面为祖国的名著还不被完全认识，甚而受到曲解，而感到深深的遗憾。还有西方学者翻译《金瓶梅》，专门摘选其中自然主义描述最为突出的篇章加以译介。一时间，西方学者好像发现了奇迹，掀起了《金瓶梅》热，说中国是“性开放的源头”，公开地在报刊上鼓吹中国要“发扬开放之传统”。还有许多资深、友善的汉学家译介中国古代的哲学著作，在把中华民族文化介绍给全世界的工作方面作出了重大贡献，但或囿于理解有误，或缘于对中国文字认识的局限，质量上乘的并不多，常常是隔靴搔痒，说不到点子上。大哲学家黑格尔曾经说过：中国有最完

备的国史。但他认为中国古代没有真正意义上的哲学，还处在哲学史前状态。这么了不起的哲学家竟然作出这样大失水准的评论，何其不幸。正如任何哲学家都要受时间、地点、条件的制约一样，黑格尔也离不开这一规律。当时他也只能从上述水平的汉学家译过去的文字去分析、理解，所以，黑格尔先生对中国古代社会的认识水平是什么状态，也就不难想象了。

中国离不开世界，世界也缺少不了中国。中国文化摄取外域的新成分，丰富了自己，又以自己的新成就输送给别人，贡献于世界。从公元5世纪开始到公元15世纪，大约有一千年，中国走在世界的前列。在这一千多年的时间里，她的光辉照耀全世界。人类要前进，怎么能不全面认识中国，怎么能不认真研究中国的历史呢?

二

中华民族是伟大的，曾经辉煌过，蓝天、白云、阳光灿烂，和平而兴旺；也有过黑暗的、想起来就让人战栗的日子，但中华民族从来是充满理想，不断追求，不断学习，渴望和平与友谊的。

中国古代伟大的思想家孔子曾经说过:“三人行，必有我师焉。择其善者而从之，其不善者而改之。”孔子的话就是要人们向别人学习。这段话正是概括了整个中华民族与人交往的原则。人与人之间交往如此，在与周边的国家交往中也是如此。

秦始皇第一个统一了中国，可惜在位只有十几年，来不及作更多的事情。汉朝继秦而继续强大，便开始走出去，了

解自己周边的世界。公元前138年，汉武帝派张骞出使西域。他带着一万头牛羊，总值一万万钱的金帛货物，作为礼物，开始西行，最远到过“安息”(即波斯)。公元前36年，班超又率36人出使西域。36个人按今天的话说，也只有一个排，显然是为了拜访未曾见过面的邻居，是去交朋友。到了西域，班超派遣甘英作为使者继续西行，往更远处的大秦国(即罗马)去访问，“乃抵条支而历安息，临西海以望大秦”(《后汉书·西域传》)。“条支”在“安息”以西，即今天的伊拉克、叙利亚一带，“西海”应是今天的地中海。也就是说甘英已经到达地中海边上，与罗马帝国隔海相望，“临大海欲渡”，却被人劝阻而未成行，这在历史上留下了遗恨。可以想见班超、甘英沟通友谊的无比勇气和强烈愿望。接下来是唐代的玄奘，历经千难万险，到“西天”印度取经，带回了南亚国家的古老文化。归国后，他把带回的佛教经典组织人翻译，到后来很多经典印度失传了，但中国却保存完好，以至于今天，没有玄奘的《大唐西域记》，印度人很难编写印度古代史。明代郑和“七下西洋”，把中华文化传到东南亚一带。鸦片战争以后，一代又一代先进的中国人，为了振兴中华，又前赴后继，向西方国家学习先进的科学思想和文明成果。这中间有我们的领导人朱德、周恩来、邓小平；有许许多多大科学家、文学家、艺术家，如郭沫若、李四光、钱学森、冼星海、徐悲鸿等。他们的追求、奋斗，他们的博大胸怀，兼收并蓄的精神，为人类社会增添了光彩。

中国文化的形成和发展过程，就是一个以众为师，以各国人民为师，不断学习和创造的过程。中华民族曾经向周边国家和民族学习过许多东西，假如没有这些学习，中华民族决不可能创造出昔日的辉煌。回顾历史，我们怎么能够不对

伟大的古埃及文明、古希腊文明、古印度文明满怀深深的感激?怎么能够不对伟大的欧洲文明、非洲文明、美洲文明、澳洲文明，以及中国周围的亚洲文明充满温情与敬意?

中华民族为人类社会曾作出过独特的贡献。在15世纪以前，中国的科学技术一直处于世界遥遥领先的地位。英国科学家李约瑟说:“中国在公元3世纪到13世纪之间，保持着一个西方所望尘莫及的科学知识水平。”美国耶鲁大学教授、《大国的兴衰》的作者保罗·肯尼迪坦言:“在近代以前时期的所有文明中，没有一个国家的文明比中国更发达，更先进。”

世界各国的有识之士千里迢迢来中国观光、学习。在这个过程中，中国唐朝的长安城渐渐发展成为国际大都市。西方的波斯、东罗马，东亚的高丽、新罗、百济、南天竺、北天竺，频繁前来。外国的王侯、留学生，在长安供职的外国官员，商贾、乐工和舞士，总有几十个国家，几万人之多。日本派出“遣唐使”更是一批接一批。传为美谈的日本人阿部仲麻吕(晁衡)在长安留学的故事，很能说明外国人与中国的交往。晁衡学成仕于唐朝，前后历时五十余年。晁衡与中国的知识分子结下了深厚的友情。他归国时，传说在海中遇难身亡。大诗人李白作诗哭悼:“日本晁卿辞帝都，征帆一片远蓬壶。明月不归沉碧海，白云愁色满苍梧。”晁衡遇险是误传，但由此可见中外学者之间在中国长安交往的情谊。

后来，不断有外国人到中国来探寻秘密，所见所闻，常常让他们目瞪口呆。《希腊纪事》(希腊人波桑尼阿著)记载公元2世纪时，希腊人在中国的见闻。书中写道:“赛里斯人用小米和青芦喂一种类似蜘蛛的昆虫，喂到第五年，虫肚子胀裂开，便从里面取出丝来。”从这段对中国古代养蚕技术的

描述，可见当时欧洲人与中国人的差距。公元9世纪中叶，阿拉伯人来到中国。一位阿拉伯作家在他所著的《中国印度闻见录》中记载了曾旅居中国的阿拉伯商人的见闻：

——一天，一个外商去拜见驻守广州的中国官吏。会见时，外商总盯着官吏的胸部，官吏很奇怪，便问："你好像总盯着我的胸，这是怎么回事？"那位外商回答说："透过你穿的丝绸衣服，我隐约看到你胸口上长着一个黑痣，这是什么丝绸，我感到十分惊奇。"官吏听后，失声大笑，伸出胳膊，说："请你数数吧，看我穿了几件衣服？"那商人数过，竟然穿了五件之多，黑痣正是透过这五层丝绸衣服显现出来的。外商惊得目瞪口呆，官吏说："我穿的丝绸还不算是最好的，总督穿的要更精美。"

——书中关于茶（他们叫干草叶子）的记载，可见阿拉伯国家当时还没有喝茶的习惯。书中记述："中国国王本人的收入主要靠盐税和泡开水喝的一种干草税。在各个城市里，这种干草叶售价都很高，中国人称这种草叶叫'茶'，这种干草叶比苜蓿的叶子还多，也略比它香，稍有苦味，用开水冲喝，治百病。"

——他们对中国的医疗条件十分羡慕，书中记载道："中国人医疗条件很好，穷人可以从国库中得到药费。"还说："城市里，很多地方立一石碑，高10肘，上面刻有各种疾病和药物，写明某种病用某种药医治。"

——关于当时中国的京城，书中作了生动的描述：中国的京城很大，人口众多，一条宽阔的长街把全城分为两半，大街右边的东区，住着皇帝、宰相、禁军及皇家的总管、奴婢。在这个区域，沿街开凿了小河，流水潺潺；路旁，葱茏的树木整然有序，一幢幢宅邸鳞次栉比。大街左边的西区，

住着庶民和商人。这里有货栈和商店，每当清晨，人们可以看到，皇室的总管、宫廷的仆役，或骑马或步行，到这里来采购。

此后的史籍对西人来华的记载，渐渐多了起来。13世纪意大利旅行家马可·波罗，尽管有人对他是否真的到过中国持怀疑态度，但他留下一部记述元代事件的《马可·波罗游记》却是确凿无疑的。这部游记中的一些关于当时中国的描述使得西方人认为是“天方夜谭”。总之，从中西文化交流史来说，这以前的时期还是一个想象和臆测的时代，相互之间充满了好奇与幻想。

从16世纪末开始，由于航海技术的发展，东西方航路的开通，随着一批批传教士来华，中国与西方开始了直接的交流。沟通中西的使命在意大利传教士利玛窦那里有了充分的体现。利玛窦于1582年来华，1610年病逝于北京，在华20余年。除了传教以外，做了两件具有历史象征意义的事，一是1594年前后在韶州用拉丁文翻译《四书》，并作了注释；二是与明代学者徐光启合作，用中文翻译了《几何原本》。

西方传教士对《四书》等中国经典的粗略翻译，以及杜赫德的《中华帝国志》等书对中国的介绍，在西方读者的眼前展现了一个异域文明，在当时及稍后一段时期引起了一场“中国热”，许多西方大思想家的眼光都曾注目中国文化。有的推崇中华文明，如莱布尼兹、伏尔泰、魁奈等，有的对中华文明持批评态度，如孟德斯鸠、黑格尔等。莱布尼兹认识到中国文化的某些思想与他的观念相近，如周易的卦象与他发明的二进制相契合，对中国文化给予了热情的礼赞；黑格尔则从他整个哲学体系的推演出发，认为中国没有真正意义上的哲学，还处在哲学史前的状态。但是，不论是推崇还

是批评，是吸纳还是排斥，中西文化的交流产生了巨大的影响。随着先进的中国科学技术的西传，特别是中国的造纸、火药、印刷术和指南针四大发明的问世，大大改变了世界的面貌。马克思说："中国的火药把骑士阶层炸得粉碎，指南针打开了世界市场并建立了殖民地，而印刷术则变成了新教的工具，变成对精神发展创造必要前提的最强大的杠杆。"英国的哲学家培根说：中国的四大发明"改变了全世界的面貌和一切事物的状态"。

三

大千世界，潮起潮落。云散云聚，万象更新。中国古代产生了无数伟大科学家：祖冲之、李时珍、孙思邈、张衡、沈括、毕升……，产生了无数科技成果：《齐民要术》、《九章算术》、《伤寒杂病论》、《本草纲目》……，以及保存至今的世界奇迹：浑天仪、地动仪、都江堰、敦煌石窟、大运河、万里长城……。但从15世纪下半叶起，风水似乎从东方转到了西方，落后的欧洲只经过400年便成为世界瞩目的文明中心。英国的牛顿、波兰的哥白尼、德国的伦琴、法国的居里、德国的爱因斯坦、意大利的伽利略、俄国的门捷列夫、美国的费米和爱迪生……，光芒四射，令人敬仰。

中华民族开始思考了。潮起潮落究竟是什么原因？中国人发明的火药，传到欧洲，转眼之间反成为欧洲列强轰击中国大门的炮弹，又是因为什么？

鸦片战争终于催醒了中国人沉睡的迷梦，最先"睁眼看世界"的一代精英林则徐、魏源迈出了威武雄壮的一步。曾国藩、李鸿章搞起了洋务运动。中国的知识分子喊出"民主

与科学”的口号。中国是落后了，中国的志士仁人在苦苦探索。但落后中饱含着变革的动力，探索中孕育着崛起的希望。“向科学进军”，中华民族终于又迎来了科学的春天。

今天，世界毕竟来到了21世纪的门槛。分散隔绝的世界，逐渐变成联系为一体的世界。现在，全球一体化趋势日益明显，人类历史也就在愈来愈大的程度上成为全世界的历史。当今，任何一种文化的发展都离不开对其它优秀文化的汲取，都以其它优秀文化的发展为前提。在近现代，西方文化汲取中国文化，不仅是中国文化的传播，更是西方文化自身的创新和发展；正如中国文化对西方文化的汲取一样，既是西方文化在中国的传播，同时也是中国文化在近代的转型和发展。地球上所有的人类文化，都是我们共同的宝贵遗产。既然我们生活的各个大陆，在地球史上曾经是连成一气的“泛大陆”，或者说是一个完整的“地球村”，那么，我们同样可以在这个以知识和学习为特征的网络时代，走上相互学习、共同发展的大路，建设和开拓我们人类崭新的“地球村”。

西学仍在东渐，中学也将西传。各国人民的优秀文化正日益迅速地为中国文化所汲取，而无论西方和东方，也都需要从中国文化中汲取养分。正是基于这一认识，我们组织出版汉英对照版《大中华文库》，全面系统地翻译介绍中国传统文化典籍。我们试图通过《大中华文库》，向全世界展示，中华民族五千年的追求，五千年的梦想，正在新的历史时期重放光芒。中国人民就像火后的凤凰，万众一心，迎接新世纪文明的太阳。

1999年8月

PREFACE TO THE *LIBRARY OF CHINESE CLASSICS*

Yang Muzhi

The publication of the *Library of Chinese Classics* is a matter of great satisfaction to all of us who have been involved in the production of this monumental work. At the same time, we feel a weighty sense of responsibility, and take this opportunity to explain to our readers the motivation for undertaking this cross-century task.

1

The Chinese nation has a long history and a glorious culture, and it has been the aspiration of several generations of Chinese scholars to translate, edit and publish the whole corpus of the Chinese literary classics so that the nation's greatest cultural achievements can be introduced to people all over the world. There have been many translations of the Chinese classics done by foreign scholars. A few dozen years ago, a Western scholar translated the title of *A Dream of Red Mansions* into "A Dream of Red Chambers" and Lin Daiyu, the heroine in the novel, into "Black Jade." But while their endeavours have been laudable, the results of their labours have been less than satisfactory. Lack of knowledge of Chinese culture and an inadequate grasp of the Chinese written language have led the translators into many errors. As a consequence, not only are Chinese classical writings widely misunderstood in the rest of the world, in some cases their content has actually been distorted. At one time, there was a "*Jin Ping Mei* craze" among Western scholars, who thought that they had uncovered a miraculous phenomenon, and published theories claiming that China was the "fountainhead of eroticism," and that a Chinese "tradition of permissiveness" was about to be laid bare. This distorted view came about due to the translators of the *Jin Ping Mei (Plum in the Golden Vase)* putting one-sided stress on the

raw elements in that novel, to the neglect of its overall literary value. Meanwhile, there have been many distinguished and well-intentioned Sinologists who have attempted to make the culture of the Chinese nation more widely known by translating works of ancient Chinese philosophy. However, the quality of such work, in many cases, is unsatisfactory, often missing the point entirely. The great philosopher Hegel considered that ancient China had no philosophy in the real sense of the word, being stuck in philosophical "prehistory." For such an eminent authority to make such a colossal error of judgment is truly regrettable. But, of course, Hegel was just as subject to the constraints of time, space and other objective conditions as anyone else, and since he had to rely for his knowledge of Chinese philosophy on inadequate translations it is not difficult to imagine why he went so far off the mark.

China cannot be separated from the rest of the world; and the rest of the world cannot ignore China. Throughout its history, Chinese civilization has enriched itself by absorbing new elements from the outside world, and in turn has contributed to the progress of world civilization as a whole by transmitting to other peoples its own cultural achievements. From the 5th to the 15th centuries, China marched in the front ranks of world civilization. If mankind wishes to advance, how can it afford to ignore China? How can it afford not to make a thoroughgoing study of its history?

2

Despite the ups and downs in their fortunes, the Chinese people have always been idealistic, and have never ceased to forge ahead and learn from others, eager to strengthen ties of peace and friendship.

The great ancient Chinese philosopher Confucius once said, "Wherever three persons come together, one of them will surely be able to teach me something. I will pick out his good points and emulate them; his bad points I will reform." Confucius meant by this that we should always be ready to learn from others. This maxim encapsulates the principle the Chinese people have always followed in their dealings with other peoples, not only on an individual basis but also at the level of state-to-state relations.

After generations of internecine strife, China was unified by Emperor

Qin Shi Huang (the First Emperor of the Qin Dynasty) in 221 B.C. The Han Dynasty, which succeeded that of the short-lived Qin, waxed powerful, and for the first time brought China into contact with the outside world. In 138 B.C., Emperor Wu dispatched Zhang Qian to the western regions, i.e. Central Asia. Zhang, who traveled as far as what is now Iran, took with him as presents for the rulers he visited on the way 10,000 head of sheep and cattle, as well as gold and silks worth a fabulous amount. In 36 B.C., Ban Chao headed a 36-man legation to the western regions. These were missions of friendship to visit neighbours the Chinese people had never met before and to learn from them. Ban Chao sent Gan Ying to explore further toward the west. According to the "Western Regions Section" in the *Book of Later Han*, Gan Ying traveled across the territories of present-day Iraq and Syria, and reached the Mediterranean Sea, an expedition which brought him within the confines of the Roman Empire. Later, during the Tang Dynasty, the monk Xuan Zang made a journey fraught with danger to reach India and seek the knowledge of that land. Upon his return, he organized a team of scholars to translate the Buddhist scriptures, which he had brought back with him. As a result, many of these scriptural classics which were later lost in India have been preserved in China. In fact, it would have been difficult for the people of India to reconstruct their own ancient history if it had not been for Xuan Zang's *A Record of a Journey to the West in the Time of the Great Tang Dynasty*. In the Ming Dynasty, Zheng He transmitted Chinese culture to Southeast Asia during his seven voyages. Following the Opium Wars in the mid-19th century, progressive Chinese, generation after generation, went to study the advanced scientific thought and cultural achievements of the Western countries. Their aim was to revive the fortunes of their own country. Among them were people who were later to become leaders of China, including Zhu De, Zhou Enlai and Deng Xiaoping. In addition, there were people who were to become leading scientists, literary figures and artists, such as Guo Moruo, Li Siguang, Qian Xuesen, Xian Xinghai and Xu Beihong. Their spirit of ambition, their struggles and their breadth of vision were an inspiration not only to the Chinese people but to people all over the world.

Indeed, it is true that if the Chinese people had not learned many

things from the surrounding countries they would never have been able to produce the splendid achievements of former days. When we look back upon history, how can we not feel profoundly grateful for the legacies of the civilizations of ancient Egypt, Greece and India? How can we not feel fondness and respect for the cultures of Europe, Africa, America and Oceania?

The Chinese nation, in turn, has made unique contributions to the community of mankind. Prior to the 15th century, China led the world in science and technology. The British scientist Joseph Needham once said, "From the third century A.D. to the 13th century A.D. China was far ahead of the West in the level of its scientific knowledge." Paul Kennedy, of Yale University in the U.S., author of *The Rise and Fall of the Great Powers*, said, "Of all the civilizations of the pre-modern period, none was as well-developed or as progressive as that of China."

Foreigners who came to China were often astonished at what they saw and heard. The Greek geographer Pausanias in the second century A.D. gave the first account in the West of the technique of silk production in China: "The Chinese feed a spider-like insect with millet and reeds. After five years the insect's stomach splits open, and silk is extracted therefrom." From this extract, we can see that the Europeans at that time did not know the art of silk manufacture. In the middle of the 9th century A.D., an Arabian writer includes the following anecdote in his *Account of China and India*:

"One day, an Arabian merchant called upon the military governor of Guangzhou. Throughout the meeting, the visitor could not keep his eyes off the governor's chest. Noticing this, the latter asked the Arab merchant what he was staring at. The merchant replied, 'Through the silk robe you are wearing, I can faintly see a black mole on your chest. Your robe must be made out of very fine silk indeed!' The governor burst out laughing, and holding out his sleeve invited the merchant to count how many garments he was wearing. The merchant did so, and discovered that the governor was actually wearing five silk robes, one on top of the other, and they were made of such fine material that a tiny mole could be seen through them all! Moreover, the governor explained that the robes he was wearing were not made of the finest silk at all; silk of the highest

grade was reserved for the garments worn by the provincial governor."

The references to tea in this book (the author calls it "dried grass") reveal that the custom of drinking tea was unknown in the Arab countries at that time: "The king of China's revenue comes mainly from taxes on salt and the dry leaves of a kind of grass which is drunk after boiled water is poured on it. This dried grass is sold at a high price in every city in the country. The Chinese call it 'cha.' The bush is like alfalfa, except that it bears more leaves, which are also more fragrant than alfalfa. It has a slightly bitter taste, and when it is infused in boiling water it is said to have medicinal properties."

Foreign visitors showed especial admiration for Chinese medicine. One wrote, "China has very good medical conditions. Poor people are given money to buy medicines by the government."

In this period, when Chinese culture was in full bloom, scholars flocked from all over the world to China for sightseeing and for study. Chang'an, the capital of the Tang Dynasty was host to visitors from as far away as the Byzantine Empire, not to mention the neighboring countries of Asia. Chang'an, at that time the world's greatest metropolis, was packed with thousands of foreign dignitaries, students, diplomats, merchants, artisans and entertainers. Japan especially sent contingent after contingent of envoys to the Tang court. Worthy of note are the accounts of life in Chang'an written by Abeno Nakamaro, a Japanese scholar who studied in China and had close friendships with ministers of the Tang court and many Chinese scholars in a period of over 50 years. The description throws light on the exchanges between Chinese and foreigners in this period. When Abeno was supposedly lost at sea on his way back home, the leading poet of the time, Li Bai, wrote a eulogy for him.

The following centuries saw a steady increase in the accounts of China written by Western visitors. The Italian Marco Polo described conditions in China during the Yuan Dynasty in his *Travels*. However, until advances in the science of navigation led to the opening of east-west shipping routes at the beginning of the 16th century Sino-Western cultural exchanges were coloured by fantasy and conjecture. Concrete progress was made when a contingent of religious missionaries, men well versed in Western science and technology, made their way to China, ushering in an era of

direct contacts between China and the West. The experience of this era was embodied in the career of the Italian Jesuit Matteo Ricci. Arriving in China in 1582, Ricci died in Beijing in 1610. Apart from his missionary work, Ricci accomplished two historically symbolic tasks — one was the translation into Latin of the "Four Books," together with annotations, in 1594; the other was the translation into Chinese of Euclid's *Elements.*

The rough translations of the "Four Books" and other Chinese classical works by Western missionaries, and the publication of Père du Halde's *Description Geographique, Historique, Chronologique, Politique, et Physique de l'Empire de la Chine* revealed an exotic culture to Western readers, and sparked a "China fever," during which the eyes of many Western intellectuals were fixed on China. Some of these intellectuals, including Leibniz, held China in high esteem; others, such as Hegel, nursed a critical attitude toward Chinese culture. Leibniz considered that some aspects of Chinese thought were close to his own views, such as the philosophy of the *Book of Changes* and his own binary system. Hegel, on the other hand, as mentioned above, considered that China had developed no proper philosophy of its own. Nevertheless, no matter whether the reaction was one of admiration, criticism, acceptance or rejection, Sino-Western exchanges were of great significance. The transmission of advanced Chinese science and technology to the West, especially the Chinese inventions of paper-making, gunpowder, printing and the compass, greatly changed the face of the whole world. Karl Marx said, "Chinese gunpowder blew the feudal class of knights to smithereens; the compass opened up world markets and built colonies; and printing became an implement of Protestantism and the most powerful lever and necessary precondition for intellectual development and creation." The English philosopher Roger Bacon said that China's four great inventions had "changed the face of the whole world and the state of affairs of everything."

3

Ancient China gave birth to a large number of eminent scientists, such as Zu Chongzhi, Li Shizhen, Sun Simiao, Zhang Heng, Shen Kuo and Bi

Sheng. They produced numerous treatises on scientific subjects, including *The Manual of Important Arts for the People's Welfare*, *Nine Chapters on the Mathematical Art*, *A Treatise on Febrile Diseases* and *Compendium of Materia Medica*. Their accomplishments included ones whose influence has been felt right down to modern times, such as the armillary sphere, seismograph, Dujiangyan water conservancy project, Dunhuang Grottoes, Grand Canal and Great Wall. But from the latter part of the 15th century, and for the next 400 years, Europe gradually became the cultural centre upon which the world's eyes were fixed. The world's most outstanding scientists then were England's Isaac Newton, Poland's Copernicus, France's Marie Curie, Germany's Rontgen and Einstein, Italy's Galileo, Russia's Mendelev and America's Edison.

The Chinese people then began to think: What is the cause of the rise and fall of nations? Moreover, how did it happen that gunpowder, invented in China and transmitted to the West, in no time at all made Europe powerful enough to batter down the gates of China herself?

It took the Opium War to wake China from its reverie. The first generation to make the bold step of "turning our eyes once again to the rest of the world" was represented by Lin Zexu and Wei Yuan. Zeng Guofan and Li Hongzhang started the Westernization Movement, and later intellectuals raised the slogan of "Democracy and Science." Noble-minded patriots, realizing that China had fallen behind in the race for modernization, set out on a painful quest. But in backwardness lay the motivation for change, and the quest produced the embryo of a towering hope, and the Chinese people finally gathered under a banner proclaiming a "March Toward Science."

On the threshold of the 21st century, the world is moving in the direction of becoming an integrated entity. This trend is becoming clearer by the day. In fact, the history of the various peoples of the world is also becoming the history of mankind as a whole. Today, it is impossible for any nation's culture to develop without absorbing the excellent aspects of the cultures of other peoples. When Western culture absorbs aspects of Chinese culture, this is not just because it has come into contact with Chinese culture, but also because of the active creativity and development of Western culture itself; and vice versa. The various cultures of

the world's peoples are a precious heritage which we all share. Mankind no longer lives on different continents, but on one big continent, or in a "global village." And so, in this era characterized by an all-encompassing network of knowledge and information we should learn from each other and march in step along the highway of development to construct a brand-new "global village."

Western learning is still being transmitted to the East, and vice versa. China is accelerating its pace of absorption of the best parts of the cultures of other countries, and there is no doubt that both the West and the East need the nourishment of Chinese culture. Based on this recognition, we have edited and published the *Library of Chinese Classics* in a Chinese-English format as an introduction to the corpus of traditional Chinese culture in a comprehensive and systematic translation. Through this collection, our aim is to reveal to the world the aspirations and dreams of the Chinese people over the past 5,000 years and the splendour of the new historical era in China. Like a phoenix rising from the ashes, the Chinese people in unison are welcoming the cultural sunrise of the new century.

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前　言

郭书春　郭金海

《四元玉鉴》[1]，三卷，中国元朝大德癸卯年（1303年）朱世杰撰，是中国传统数学水平最高的著作。

（一）

为了了解《四元玉鉴》在中国和世界数学史上的地位，我们首先需要简要介绍一下《四元玉鉴》之前中国传统数学的概况。

中国传统数学是中国古代最为发达的基础学科之一，自公元前3世纪到14世纪初领先于世界先进水平约1700余年，是此时世界数学发展的主流。

中国有文字记载的历史相当早，然而夏、商、西周三代和春秋时期没有数学著作流传到今天，数学发展情形不十分清楚。现在所知，完成当时世界上最方便的记数制度——十进地位制记数法，创造当时世界上最先进的计算工具——算筹，是两项有世界意义的成就。中国传统数学与古希腊数学具有不同的形态，长于计算，与此两项创造有密不可分的关系。

战国、秦、汉奠定了中国作为一个统一国家在体制、事功、疆域、物质文明和思想文化等方面的基础，奠定了中华民族的文化心理结构，也奠定了中国传统数学的基础。战国到西汉初年，《算数书》[2]、《周髀算经》[3]、《九章算术》[4]等成书，在分数四则运算、比例和比例分配算法、盈不足算法、开方法、线性方程组解法、正负数加减法则、解勾股形和勾股数组等方面走在了世界的前面，有的超前其他文化传统

数百年，甚至上千年。中国传统数学达到第一个高潮。《九章算术》确立了中国传统数学的基本框架，具有理论密切联系实际的风格，以计算为中心，并且算法具有构造性、机械化的特点，不仅影响了此后约二千年间中国和东方的数学发展，而且标志着中国（还有后来的印度和阿拉伯地区）取代地中海沿岸的古希腊成为世界数学研究的中心，标志着以研究数量关系为主、以归纳逻辑与演绎逻辑相结合的算法倾向取代以研究空间形式为主、演绎逻辑的公理化倾向成为世界数学发展的主流。

东汉末年至魏晋南北朝，庄园农奴制占据经济政治舞台的中心，思想界以谈“三玄”(《周易》、《老子》、《庄子》)为主的辩难之风取代了烦琐的两汉经学，中国社会进入一个新的阶段。数学知识的积累，以及思想界辩难之风和墨家思想的影响，促使魏刘徽撰《九章算术注》(公元263年)。他“析理以辞，解体用图”，提出了许多严格的数学定义，以演绎逻辑为主要方法全面证明了《九章算术》的算法，奠定了中国传统数学的理论基础。他对圆面积公式和刘徽原理[5]的证明在世界数学史上首次将极限思想和无穷小分割方法引入数学证明，其割圆术和“求微数”的思想奠定了中国的圆周率计算领先世界数坛千余年的基础，而解决多面体体积理论的刘徽原理实际上在考虑20世纪数学大师希尔伯特(1862—1943)的第三问题(1900年)所涉及的内容[6]。刘徽的《海岛算经》[7]将以重差术为主的中国测望技术发展得相当完善，以致在明末西方测望方法传入之前，1300年间没有大的改进。[8]这表明中国数学进入了第二个高潮。南朝祖冲之(429—500)的《缀术》是一部更高深的著作，可惜隋唐算学馆的学官“莫能究其深奥，是故废而不理”[9]，遂失传。目前我们知道的只是祖冲之父子在刘徽基础上推进的两项成就：将圆周率精确到8位有效数字，并提出密率 $\frac{355}{113}$；他和他的儿子提出祖暅之原理(等价于西方的卡瓦列利原理)，彻底解决了球体积问题。这

一时期还提出了一次同余式解法（《孙子算经》[10]，约公元400年）、百鸡术（《张丘建算经》[11]，5世纪）等新的研究方向。

自唐中叶起，随着农业、手工业和商业的大发展，中国的经济、政治出现许多新的因素，到宋元时期，发展得更加成熟，同时，思想界还比较宽松。中国古代科学技术发展到一个新的高峰。北宋贾宪撰《黄帝九章算经细草》[12]（11世纪上叶），进一步抽象《九章算术》的算法，创造"开方作法本源"即贾宪三角以及"增乘开方法"，开创了宋元数学的高潮。

13世纪是中国传到今天的重要数学著作最多的时期。当时有南、北两个数学中心。南宋统治的长江下游，出现了以秦九韶、杨辉为代表的以研究高次方程解法、同余式组解法、改进乘除捷算法为主的中心。秦九韶（约1202—1261）撰《数书九章》[13]（1247年），提出"大衍总数术"，从而完善了一次同余式组解法，他还改进了高次方程的数值解法。杨辉撰《详解九章算法》[14]（1261年）、《杨辉算法》[15]（1274—1275）等，在垛积术、改进乘除捷算法等方面有成绩。金元统治下的北方数学中心则发展了列高次方程的方法"天元术"，解二元、三元高次方程组的"二元术"、"三元术"，勾股容圆知识和垛积术。李冶（1192—1279）的《测圆海镜》[16]（1248年）、《益古演段》[17]（1259年）都是使用天元术的著作。王恂（1235—1281）、郭守敬（1231—1316）在《授时历》中使用了垛积招差法。13世纪末的朱世杰在元朝统一中国之后，汲纳了南、北两个数学中心的长处，在数学造诣上达到了新的高度。

（二）

朱世杰，字汉卿，号松庭，燕（一作燕山，今北京或其附近地区）人，生、卒年及生平不详。莫若说他"以数学名家周游湖海二十余年"，

“四方之来学者日众”。[18] 1299年、1303年先后在扬州刊刻了《算学启蒙》[19]、《四元玉鉴》。他是一位中国历史上少见的职业数学家和数学教育家。

《算学启蒙》卷首有预备知识，然后分三卷，20门，259问，包括了从乘除运算及其捷算法到开方术、天元术、方程术、垛积术等当时数学各方面的内容，由浅入深，形成了较完整的系统。它应该是朱世杰教授学子的教材。改进筹算的乘除法是宋元数学的重要方面。这是自唐中叶起随着商业的繁荣而自发的民间活动，因而为李冶等大数学家所不屑。杨辉、朱世杰等则对之十分重视。朱世杰在《算学启蒙》中所整理的各种乘除捷算法口诀比杨辉的更加完整、流畅，许多口诀与现代的珠算口诀完全一致。乘除捷算法的发展导致珠算盘的产生，珠算盘在明代最终取代了算筹，至今在中国、日本和东南亚地区发挥着有益的作用。

《四元玉鉴》，三卷，24门，288问。卷首是今古开方会要之图等四种图，以及以四个题目示范天元术、二元术、三元术、四元术的解法的四象细草假令之图。提出“四元术”即多元高次方程组解法，将高阶等差级数求和问题和高次招差法发展到相当成熟的程度，是《四元玉鉴》的主要成就。

乾隆年间修《四库全书》，《算学启蒙》、《四元玉鉴》因未发现而没有收入。清嘉庆初年，阮元（1764—1849）在浙江访得《四元玉鉴》。此后李锐（1768—1817）、沈钦裴[20]、罗士琳[21]（1789—1853）、戴煦[22]（1805—1860）、李善兰[23]（1811—1882）、丁取忠[24]、华蘅芳[25]（1833—1902）、周达[26]等都参与了《四元玉鉴》的研究，在四元消法和垛积招差法等方面做了有意义的工作。

（三）

高次方程的数值解法是中国传统数学最发达的一个分支。在中国古代，“方程”指今之线性方程组，今之求解一元方程的方法，则都称为“开方术”。《九章算术》少广章的开平方术、开立方术，是世界数学史上最早的开方程序。这种方法经过刘徽、《孙子算经》等的改进，到祖冲之[27]、王孝通[28]（7世纪）已能解二次、三次方程，祖冲之还能解含有负系数的方程。而贾宪将传统的开方术推广到开高次方，称为“立成释锁法”，并创造贾宪三角作为其“立成”。贾宪还创造了增乘开方法[29]，以随乘随加代替使用贾宪三角的系数。秦九韶、李冶、朱世杰等则以增乘开方法为主导，提出正负开方术，是完整的求解高次方程正根的方法。当求出根的整数部分后出现开方不尽的情形时，朱世杰使用连枝同体术（又称为之分法），将减根方程进行变换，得出其分数部分。

宋元之前，列方程没有统一的程式，为了列出方程，针对不同的问题往往需要构思巧妙的方法。金元时期，数学家们创造了天元术。所谓天元术就是设未知数列方程的方法：先设某某为天元一，然后根据问题的条件，列出两个等价的天元式即多项式，两天元式相消，便得到一个一元方程。这个过程称为如积相消。

祖颐说：“平阳蒋周撰《益古》，博陆李文一撰《照胆》，鹿泉石信道撰《钤经》，平水刘汝谐撰《如积释锁》，绛人元裕细草之，后人始知有天元也。”说明天元术的产生、发展、完善有一个过程。李冶还见到过以天、地分别表示天元的正、负幂的方法[30]。李冶取消了表示负幂的地元，只用天元，并依据它与表示常数项的“元”或“太”字的相对位置确定其幂次。李冶先是在《测圆海镜》中采取天元的正幂在上、负幂在下的方式，后在《益古演段》中则颠倒过来，遂成为天元术的通用

表示方式。朱世杰在《算学启蒙》、《四元玉鉴》中都是采取后者。

可能受到《九章算术》方程术的启发，人们将天元术推广到二元、三元、四元的情形，就成为二元术、三元术、四元术，即分别是二元、三元、四元高次方程组的解法。祖颐在谈到这个过程时说："平阳李德载因撰《两仪群英集臻》，兼有地元，霍山邢先生颂不高弟刘大鉴润夫撰《乾坤括囊》，末仅有人元二问。吾友燕山朱汉卿先生演数有年，探三才之赜，索《九章》之隐，按天、地、人、物立成四元。"[31] 可见，四元术的发明者是朱世杰。

莫若说：四元式的布置，"其法以元气居中，立天元一于下，地元一于左，人元一于右，物元一于上。"[32] 二元术有二个二元式，三元术有三个三元式，四元术有四个四元式。设天、地、人、物四元分别为 x, y, z, u，则四元式的表示法就是：

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots y^3u^3 & y^2u^3 & yu^3 & u^3 & u^3z & u^3z^2 & u^3z^3 \cdots \\
\cdots y^3u^2 & y^2u^2 & yu^2 & u^2 & u^2z & u^2z^2 & u^2z^3 \cdots \\
\cdots y^3u & y^2u & yu & u & uz & uz^2 & uz^3 \cdots \\
\cdots y^3 & y^2 & y & \text{太} & z & z^2 & z^3 \cdots \\
\cdots xy^3 & xy^2 & xy & x & zx & z^2x & z^3x \cdots \\
\cdots x^2y^3 & x^2y^2 & x^2y & x^2 & zx^2 & z^2x^2 & z^3x^2 \cdots \\
\cdots x^3y^3 & x^3y^2 & x^3y & x^3 & zx^3 & z^2x^3 & z^3x^3 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

相邻两元的幂次的积置于相应行列的交叉处，不相邻诸元的幂次之积置于夹缝中。

四元消法即多元高次方程组的消去法是《四元玉鉴》的突出创造，也就是将四元消成三元，再消成二元，最后化成一元高次方程，用正负开方术求解。然而，如何消法，朱世杰在"四象细草假令之图"的"三

才运元"、"四象会元"中只说"剔而消之"、"人易天位"或"物易天位"，在"两仪化元"间及将"三才运元"、"四象会元"间化成二元二式之后只说"互隐通分相消"，甚为简括。祖颐概括说："上升下降，左右进退，互通变化，乘除往来，用假象真，以虚问实，错综正负，分成四式。必以寄之、剔之，余筹易位，横冲直撞，精而不杂，自然而然，消而和会，以成开方之式也。"[33] 人们仍不得要领。清中叶以来研究四元消法的各家中，一般认为沈钦裴的理解比较符合原意。

"剔而消之"是三元式和四元式的消元法，亦即将全式剔而为二。以三元式消去地元为例。以"太"所在一行将两个全式分别剔而为二，以地元除第一式的左半，然后乘第二式的右半，以地元除第二式的左半，然后乘第一式的右半，将所得二式相消，即可降低其次数。反复运用，最后消去地元。如：

$$A_2y^2 + A_1y + A_0 = 0$$

$$B_2y^2 + B_1y + B_0 = 0 \qquad (1)$$

其中A_2，A_1，A_0，B_2，B_1，B_0是不含y而仅含有x，z的多项式。欲消去末项，先将以上二式改写成

$$(A_2y + A_1)y + A_0 = 0$$

$$(B_2y + B_1)y + B_0 = 0$$

剔而消之就是以二式中的A_0，B_0互乘两式，相消得到

$$(A_2B_0 - A_0B_2)y + (A_1B_0 - A_0B_1) = 0 \qquad (2)$$

同样，欲消去首项，以（1）中的A_2，B_2互乘两式，相消得到

$$(A_1B_2 - A_2B_1)y + (A_0B_2 - A_2B_0) = 0 \qquad (3)$$

将（2）、（3）式分别与（1）之一式联立，施用同样的程序可以得到

$$C_1y + C_0 = 0 \qquad (4)$$

同样，C_1，C_0是不含y而仅含有x，z的多项式。对（2）、（4）式或

(3)、(4)式施用同样的程序，便消成不含y而仅含有x，z的方程。

人易天位，是四元消法中完成“剔而消之”之后进行的变换。在三元式中，若消去的是地元，则需将所余的天、人二元式以“太”为中心转90度，使二元式由第四象限转至第三象限。同样，物易天位是在四元式中，若消去的是天元，则需将所余的地、人、物的三元式以“太”为中心转180度，使三元式由第一、二象限转至第三、四象限。这种变换并不改变方程。之所以做这种变换，是因为在筹算中人们常用第三象限。

互隐通分是将二元多行式消成2行所使用的方法。如关于x，y的n行式

$$A_nx^n + \cdots + A_2x^2 + A_1x + A_0 = 0$$

$$B_nx^n + \cdots + B_2x^2 + B_1x + B_0 = 0$$

其中系数A_i，B_i，$i = 1, 2, 3, \cdots, n$是不含x，只含有y的多项式。通过互乘相消或先消去末项，或先消去首项，最后消成2行二式方程。[34]

朱世杰的消去法对现代数学的研究有极大的启迪作用。吴文俊从《九章算术》方程术的消元法和朱世杰的四元消法中得到启示，结合现代数学的某些理论，发现了三角化整序法，“据之以得出了彻底解决高次联立方程组求解的方法”。[35]

朱世杰用天元术、二元术、三元术、四元术解决的问题中，常常借助于垛积术即高阶等差级数求和法与招差术提出某些假设，因此，垛积术与招差术成为《四元玉鉴》的另一重要成就。

《九章算术》中有等差级数的求和公式，《张丘建算经》的等差级数知识更为完整一些。宋元时期农业和手工业发达，生产的大量的禾捆、水果、坛、罐、瓶、砖瓦等垛成堆垛，需要计算其数量，垛积术应运而生。

北宋科学家沈括(1031—1095)首先注意到垛成《九章算术》刍童形状的堆垛因“积而有隙”，不能用刍童公式求解，从而创造了“隙积术”，[36]实际上是一种二阶等差级数的求和公式。南宋杨辉给出了另外

几个垛积公式，也都是二阶等差级数求和问题。王恂、郭守敬等用垛积招差术解决了《授时历》中的日、月的日行度数的计算问题。朱世杰则将垛积招差术发展到更高的阶段。

朱世杰在《四元玉鉴》中使用了一串三角垛的求和公式。[37] 所谓三角垛就是底面为三角形的堆垛。其通项是朱世杰娴熟的茭草垛即自然数列的前 n 项之和：

$$S_n=\sum_{i=1}^{n} i=1+2+\cdots+n=\frac{1}{2!}n(n+1);$$

三角垛（或落一形垛）的求和公式是：

$$S_n=\sum_{i=1}^{n}\frac{1}{2!}i(i+1)=\frac{1}{3!}n(n+1)(n+2),$$

这实际上是二阶等差级数求和问题，是杨辉在《详解九章算法》商功章鳖臑的比类中提出的。朱世杰又使用了撒星形垛（或三角落一形垛）：

$$S_n=\sum_{i=1}^{n}\frac{1}{3!}i(i+1)(i+2)=\frac{1}{4!}n(n+1)(n+2)(n+3);$$

三角撒星形垛（或撒星更落一形垛）：

$$S_n=\sum_{i=1}^{n}\frac{1}{4!}i(i+1)(i+2)(i+3)$$
$$=\frac{1}{5!}n(n+1)(n+2)(n+3)(n+4);$$

三角撒星更落一形垛：

$$S_n=\sum_{i=1}^{n}\frac{1}{5!}i(i+1)(i+2)(i+3)(i+4)$$
$$=\frac{1}{6!}n(n+1)(n+2)(n+3)(n+4)(n+5)。$$

这几个公式中，前一个的前 n 项之和恰是后一个的第 n 项，故后一个被称为前一个的落一形垛。同时，上述各堆垛的诸项依次是贾宪三角的第 2，3，4，5，6 条斜线上的各数，而其和恰恰是第 3，4，5，6，7 条斜线上的第 n 个数。这就是为什么朱世杰用两组平行于左右两斜的平行线将贾宪三角的各个数连结起来。可见，这些内容在《四元玉鉴》中似乎没有条理，实际上在朱世杰的头脑中已经形成完整的体系。因此，朱世杰实际上已经掌握了以 $\frac{1}{p!}i(i+1)(i+2)\cdots(i+p-1)$ 为通

项的一串三角垛的前 n 项和的公式：

$$S_n=\sum_{i=1}^{n}\frac{1}{p!}\ i(i+1)(i+2)\cdots(i+p-1)$$
$$=\frac{1}{(p+1)!}\ n(n+1)(n+2)\cdots(n+p)。$$

显然，当 $p=1,2,3,4,5$ 时,分别是上述各种垛积。

朱世杰又使用了岚峰形垛的求和公式。岚峰形垛是以三角垛的各项再乘以该项的项数即以 $\frac{1}{p!}\ i(i+1)(i+2)\cdots(i+p-1)\ i$ 为通项的垛积，其前 n 项和的公式是：

$$S_n=\sum_{i=1}^{n}\frac{1}{p!}\ i(i+1)(i+2)\cdots(i+p-1)\ i=\frac{1}{(p+2)!}\ n$$
$$(n+1)(n+2)\cdots(n+p)[(p+1)n+1]$$

当 $p=1,2,3,\cdots$，时，分别是四角垛、岚峰形垛、三角岚峰形垛或岚峰更落一形垛……。

此外，朱世杰还解决了三角台垛、四角台垛等更复杂的垛积。

朱世杰在“如像招数”门中使用了招差公式：

$$f(n)=n\Delta_1+\frac{1}{2!}n(n-1)\Delta_2+\frac{1}{3!}n(n-1)(n-2)\Delta_3$$
$$+\frac{1}{4!}n(n-1)(n-2)(n-3)\ \Delta_4。$$

其中 Δ_1，Δ_2，Δ_3，Δ_4 是上差、二差、三差、四差，二差的系数是以 $n-1$ 为底子的茭草垛积，三差的系数是以 $n-2$ 为底子的三角垛积，四差的系数是以 $n-3$ 为底子的撒星形垛积。这一公式与现代通用的形式完全一致。[38]可见，招差术是通过求差运算将垛积分解为诸差乘三角垛的积，“诸差”由所给垛积惟一确定。[39]

朱世杰的这些成就大多超前其他文化传统几个世纪，有的是欧洲17、18、19世纪的数学大师们才解决的。

在《四元玉鉴》之后，中国数学一落千丈，出现了明代大数学家看不懂宋元的增乘开方法、天元术、四元术等重要数学成就，汉魏南北朝宋元的重要数学著作无人问津，甚至失传的可悲局面。阿拉伯和西方数学先后超过了中国，中国失去了数学大国的地位。因而《四元玉鉴》是

中国传统数学繁荣时期最后一部有创造性的著作，也是水平最高的著作。

（四）

本书中文部分的《四元玉鉴》本文以丁取忠光绪二年（1876年）在罗士琳校勘基础上的重校本[40]为底本，由郭书春译成现代汉语，并做了必要的校勘和注释。

本书的英文部分主要采用了中国科学院自然科学史研究所图书馆馆藏陈在新（1879—1945）的英译稿。陈在新，字化民，河北宛平县南辛庄（今北京海淀香山附近）人，1901年毕业于北京汇文大学，并留校教数学。后留美，1912年在哥伦比亚大学获得硕士学位。他获得博士学位的时间约于1913年前，很可能是中国第一位留美数学博士。他回国后仍执教于汇文大学。该校于1919年并入刚成立的燕京大学，1920年陈在新应校长司徒雷登的聘请任数学系主任，至1936年因罹病而辞职。他重视数学史的教学与学习，在执掌燕京大学数学系时，开设了数学史和中国数学史两门课程。这在今天的大学数学系中，也是少见的。

陈在新还在哥伦比亚大学时，受到数学史家史密斯（D. E. Smith，1860—1944）博士的鼓励，开始了《四元玉鉴》的英译；后来在他的同事寇恩慈（E. L. Konantz）的帮助下，大约在1925年底完成了这项工作。在某种意义上说，这是两人合作的产物。他们的译稿由燕京哈佛学社于1929年交给著名科学史家萨顿（George Sarton，1884—1956）审阅，萨顿给予了高度评价，建议在文字上做几处必要的修改后出版，不知何故未果。1940年萨顿再次见到此译稿，希望作为燕京哈佛系列之一出版，[41]亦未付梓。陈在新的译稿和注释表明，他对四元消法及朱世杰处理某些类型问题的方法比如连枝同体术有相当深入的理解。他的工作对当前的中国数学史研究，对西方了解中国传统数学具有重要的参考价值。

郭金海对英译稿进行了整理。原稿卷中"明积演段"门第14至20题，"拨换截田"门第13题遗失，郭金海做了补译，并将郭书春补充的注释及此前言译成英文，将陈在新的序译成中文，查询了陈在新的有关资料，做了全部的中英对照的整理工作。陈在新的注释标以（陈或C），郭书春的补注标以（郭或G）。此外，陈在新原稿列出的开方式中都在常数项旁标以"太"字，这是清中叶以来传承已久的一个误解。实际上，将"太"置于常数项旁，或将"元"置于一次项旁，所表示的是多项式，而不是开方式（即今天所说的一元方程）。在大元术中，两个等价的天元式通过"如积相消"，一经化成开方式，便不再保留"太"或"元"字。[42] 因此，我们删去了陈在新的注释中的这类"太"或"元"字。对汉字的注音，陈在新都用罗马化文字音标，除"陈在新"的名字外我们一律改为现代汉语拼音。因此，陈在新原稿中的"罗马化文字的发音"，我们没有收入。

这项工作得到中国科学院自然科学史研究所和辽宁教育出版社领导的支持，得到国家图书馆、自然科学史研究所图书馆和陈在新之孙女陈懿德、他的同事徐献瑜以及许苏葵、道本周（Joseph W. Dauben）、刘钝、林文照、李兆华、梁良兴、田淼、徐义保等先生的帮助，特在此表示衷心感谢。尤其是道本周、林文照、李兆华先生对本前言提出了宝贵的修正意见，道本周与梁良兴、徐义保先生修改了其英译稿，补充了外文研究资料，他们的高风亮节，使本书得益良多。

【注释】

[1]〔元〕朱世杰：四元玉鉴。见：郭书春主编：中国科学技术典籍通汇·数学卷。第1册第1205~1275页。郑州：河南教育出版社，1993年。该书书名陈在新译作 *Precious Mirror of the Four Elements*，李俨、杜石然所著《中国数学简史》的英译本［John N. Crossley（郭树理）与 Anthony W.-C. Lun（伦华祥）译：*Chinese Mathematics. A Concise History*，Oxford: Clarendon Press, 1987，P. 111］采用陈在新的译法；Jean-Claude Martzloff（马若安）所著 *Histoire des mathématiques chinoises*

(《中算史导论》的英译本 *A History of Chinese Mathematics* (Stephen S. Wilson 译, Berlin: Springer-Verlag, 1997, p. 17)译作 *Jade Mirror of the Four Origins*。而马若安认为,书名最好译作"Mirror [trustworthy as] jade [relative to the] four origins [unknowns]"。见上述马若安的著作,第153页。由于本书书名中的"玉"的本义是"jade","元"的本义是"element",而从该书的上下文来看,"元"的意思是"unknown"。因此,我们根据道本周(Joseph W. Dauben)的建议改译为 *Jade Mirror of the Four Unknowns*。值得注意,在数学意义上,"元"指"unknown"。本书有时用此译法,或者译作"unknown element"。关于朱世杰及其数学知识的详尽研究,可参考:钱宝琮主编:中国数学史。第175~205页。北京:科学出版社,1964年;郭书春、刘钝主编:李俨钱宝琮科学史全集。第5卷第197~277页。沈阳:辽宁教育出版社,1998年;杜石然:朱世杰研究。见:钱宝琮等:宋元数学史论文集。第166~209页。北京:科学出版社,1966年;John Hoe(谢元作)著:*Les systèmes d'équations polynômes dans le Siyuan Yujian (1303)*(《〈四元玉鉴〉中的多项式方程组》),Paris: Collège de France, Institut des Hautes Études Chinoises (Mémoires de l'institut des Hautes Études Chinoises, vol. 6);Andrea Bréard(白安雅):*Re-kreation eines mathematischen Konzeptes im cinesischen Diskurs. "Reihen" vom 1. bis zum 19. Jahrhundert* [《中国语境中数学概念的再创造(从1世纪到19世纪)》], Boethius, vol. 42. Stuttgart: Franz Steiner Verlag, 1999, pp. 178-264。

[2]《算数书》是20世纪80年代湖北张家山汉墓出土的数学竹简。见:张家山汉墓竹简整理小组:张家山汉墓竹简[二四七号墓]。北京:文物出版社,2002年。郭书春:《筭数书》校勘。《中国科技史料》,第22卷(2001年),第3期,第202~219页。

[3][西汉]周髀算经,赵爽注,刘钝、郭书春点校。见:算经十书,郭书春、刘钝点校。辽宁教育出版社,1998年;台北:九章出版社,2001年。第29~80页。亦见英译本:Christopher Cullen(古克礼):*Astronomy and Mathematics in Ancient China, the Zhou Bi Suan Jing*(《中国古代的天文学与数学:周髀算经》)。Cambridge: Cambridge University Press, 1996。

[4]〔西汉〕九章算术，刘徽注，李淳风等注释，郭书春汇校。沈阳：辽宁教育出版社，1990年；增补版，辽宁教育出版社，台湾九章出版社，2004年。又：郭书春点校。见：算经十书，郭书春、刘钝点校。辽宁教育出版社，1998年；台北：九章出版社，2001年。第81～244页。亦见英译本：Shen Kangshen（沈康身），John Crossley（郭树理），Anthony W.-C. Lun（伦华祥）：*The Nine Chapters on the Mathematical Art: Companion and Commentary*（《〈九章算术〉导读与注释》）。Oxford: Oxford University Press and Beijing: Science Press, 1999。中法对照本：Karine Chemla（林力娜），Guo Shuchun（郭书春）：*Les Neuf Chapitres：le classique mathèmatique de la Chine Ancienne*（《九章算术：中国古代的数学经典》），Paris：Dunod Editeur，2004。

[5]刘徽原理是：将一长方体斜解成两楔形体，称为堑堵。将一堑堵斜解成一阳马（直角四棱锥）与一鳖臑（四面均为勾股形的四面体），则阳马与鳖臑的体积之比恒为2∶1。这是刘徽多面体体积理论的基础。

[6]希尔伯特：数学问题——在1900年巴黎国际数学家大会上的讲演。见：*Göttinger Nachrichten*（1900），pp. 253-297；Mary F. Winston将其译为英文：*Mathematical Problems: Lecture Delivered Before the International Congress of Mathematicians at Paris in 1900*，见：*Bulletin of the American Mathematical Society*, vol. 8（1902），pp. 437-479；再版于David Hilbert, *Gesammelte Abhandlungen*. Berlin: Springer-Verlag, 1935, vol. 3，pp. 290-329；Felix Brouder编：*Mathematical Developments Arising from Hilbert Problems*。Providence, RI: American Mathematical Society, 1976. 亦被李文林、袁向东翻译为中文，可参见：数学史译文集。上海：上海科技出版社，1981年，第60～84页。

[7]〔魏〕刘徽：海岛算经，李淳风等注释，郭书春点校。见：算经十书，郭书春、刘钝点校。辽宁教育出版社，1998年；台北：九章出版社，2001年。第245～255页。亦见Frank J. Swetz的英译本：*The Sea Island Mathematical Manual: Surveying and Mathematics in Ancient China.* University Park: Pennsylvania State University Press, 1992; Shen Kangshen（沈康身），John Crossley（郭树理），Anthony W.-C. Lun（伦华祥）：*The Nine Chapters on the Mathematical Art: Companion and Commentary*《〈九章算术〉导读与注释》。Oxford: Oxford University Press and Beijing: Science Press, 1999，pp. 518-559。

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INTRODUCTION

by Guo Shuchun and Guo Jinhai

The *Si Yuan Yu Jian* (Jade Mirror of the Four Unknowns)[1] was written in three parts by Zhu Shijie and published in 1303 during the Yuan dynasty. It is the most advanced work of traditional Chinese mathematics.

1

In order to understand its status in the history of mathematics in China and internationally, we first need to introduce briefly the general situation of traditional Chinese mathematics before the *Jade Mirror of the Four Unknowns* was written.

Traditional Chinese mathematics is one of the most advanced basic disciplines in ancient China, and from the 3rd century BCE to the beginning of the 14th century CE, was the leading and most advanced in the world for more than 1700 years. Compared with the state of mathematics in the medieval West, the Jade Mirror certainly represents one of the leading and most significant mathematical developments on the world mathematical scene.

Although Chinese history written in characters is very early, no mathematical works have survived from three of the earliest dynasties, including the Xia, Shang, and Western Zhou, nor from the Spring and Autumn period as well. Therefore, the nature of the development of Chinese mathematics during these periods is not clearly understood. At present, we know that the Chinese created the most convenient means of numeration in the world: the

decimal place-value system, accompanied by what at the time was the most advanced calculation method as well: counting-rods. These are two of the world's most significant accomplishments. The character of traditional Chinese mathematics is different from ancient Greek mathematics, but the number system and method of computation are two innovations that have an inseparable connection.

The Warring States period and the Qin and Han dynasties established the foundations for unifying China with respect to systems of organization, business practices, territorial boundaries, cultural products, intellectual matters, etc. Furthermore, they established the foundations for Chinese civilization and also for traditional Chinese mathematics. The *Suan Shu Shu* (Book of Arithmetic)[2], the *Zhou Bi Suan Jing* (Arithmetical Classic of the Gnomon and the Circular Paths of Heaven)[3], and the *Jiu Zhang Suan Shu* (The Nine Chapters of Mathematical Procedures)[4], were finished between the Warring States period and the beginning of the Western Han dynasty. Many aspects of Chinese mathematics were ahead of the world's leading levels, and some exceeded them for more than a thousand years, including computations with the four basic arithmetic operations for fractions, the arithmetic of proportion and proportional distribution, the rule of excess and deficiency, the approximation of square and cube roots, solutions of simultaneous equations, rules of addition and subtraction for positive and negative numbers, finding so-called "Pythagorean" triples, etc. *The Nine Chapters of Mathematical Procedures* established the basic framework for traditional Chinese mathematics. It closely relates theory and practice. Focusing on computations, the character of its arithmetic is structural and mechanical, which influenced the development of Chinese and oriental mathematics for the next two thou-

sand years or so thereafter. It also shows that alongside the significant mathematical developments in the Mediterranean, China (and later India and the Islamic Near East) were also important centres of mathematical activity. Chinese mathematics also represents a different emphasis, one employing the relationships between quantities rather than geometric space and logical deduction. It reflects as well mathematics turning away from an emphasis primarily upon deductive logic to an arithmetic tendency combining inductive and deductive logic.

From the end of the Eastern Han dynasty to the Wei, Jin and Southern and Northern dynasties, economy and politics were based upon a system of manorial serfdom. The practice among scholars of debating the "three mysteries" (these were the *san xuan*, which mainly included the *Book of Changes*, the *Lao Zi* and *Zhuang Zi*) replaced the trivial *jing xue* (classical studies) in the two Han periods. Chinese society developed into a new phase. The accumulation of mathematical knowledge, the spirit of debate among scholars, and the influence of Mohist thought spurred Liu Hui of the Wei period to write his *Commentary on "The Nine Chapters of Mathematical Procedures"*. He "analyzed the principles, and used the diagrams to display their components," and gave many rigorous mathematical definitions. He used deductive logic as his main method to prove the mathematical principles of *The Nine Chapters of Mathematical Procedures*, the most comprehensive work of traditional Chinese mathematics. His proof of the formula for the area of a circle and Liu Hui's theorem [5] were the first in China to introduce the concept of limit and the method of infinitesimals into mathematical proofs. His method of cutting the circle without limit and his idea of "solving with infinitesimals" (*qiu wei shu*) provided the foundation for the calcula-

tion of π in Chinese mathematics. In fact, Liu Hui's theorem for determining the volumes of solids is related to the third problem brought forward by David Hilbert (1862—1943), who presented his famous list of 23 "unsolved problems for the 20th century" at the International Congress of Mathematicians held in Paris in 1900.[6] Liu Hui's *Hai Dao Suan Jing* (Sea Island Mathematical Manual)[7] was devoted to Chinese techniques for measuring heights and distances, for which the method of double differences was the main method used, one that was not surpassed for more than 1300 years, until the introduction of Western methods for the measurement of heights and distances at the end of the Ming dynasty.[8] Another landmark contribution to Chinese mathematics was the *Zhui Shu* by Zu Chongzhi (429—500 CE), a more profound book written in the Southern dynasty. It is a pity that officers at the Suan Xue Guan (Mathematics Bureau) during the Sui and Tang dynasties were unable to understand it, and as a result, it was eventually abolished.[9] Subsequently, the book was lost. At present, we only know two of its contributions advanced by Zu Chongzhi and his son Zu Gengzhi based on the foundations of Liu Hui's achievement. One of these was the computation of the value of π to eight decimal places, as well as a more accurate ratio for π : $\frac{355}{113}$; the second was Zu Gengzhi's theorem (equivalent to Cavalieri's theorem in the West), by which Zu Gengzhi determined the correct formula for the volume of a sphere. Other classic texts of ancient Chinese mathematics gave solutions for expressions of congruence of first degree, as did the *Sun Zi Suan Jing* (Master Sun's Mathematical Manual) (4th century CE)[10], and the method of the "Hundred Fowls" in the *Zhang Qiujian Suan Jing* (Zhang Qiujian's Mathematical Classic) (5th century CE)[11], among others.

With the great development of agriculture, handicraft industry, and commerce, many new factors influenced Chinese economy and politics beginning in the middle period of the Tang dynasty. During the Song and Yuan dynasties, these influences became more mature. Moreover, the atmosphere within the circle of intellectuals was comparatively free. Ancient Chinese science and technology developed to new heights. In the first half of the 11th century, Jia Xian composed the *Huang Di Jiu Zhang Suan Jing Xi Cao* (Detailed Explanation of the Ancient Classic, Nine Chapters) (first half of 11th century CE)[12], in which he made the solution methods of *The Nine Chapters of Mathematical Procedures* more abstract. He created the method of *kai fang zuo fa ben yuan* (origin of the method for extracting roots), that is, Jia Xian's triangle and the method of *zeng cheng kai fang fa* (a method for extracting roots using addition and multiplication, similar to Horner's method), which represents the apex of Chinese mathematics in the Song and Yuan dynasties.

The 13th century was a period during which the most important mathematical works were handed down to the present. At that time, there were two mathematical centres, one in the south, another in the north. The lower reaches of the Yangtze River were governed by the Southern Song dynasty, where a centre emerged which included Qin Jiushao and Yang Hui. They advanced solutions for equations of higher powers, for linear indeterminate equations, and improved the simple methods of calculation for multiplication and division among their main contributions. Qin Jiushao (ca. 1202 — 1261 CE) composed the *Shu Shu Jiu Zhang* (Mathematical Treatise in Nine Sections) (1247)[13], in which he put forward the method *da yan zong shu shu* (general *da yan* method), thereby consummated the method for expressions of con-

gruence of first degree, and improved the numerical solutions of equations of higher powers. Yang Hui composed the *Xiang Jie Jiu Zhang Suan Fa* (Detailed Explanations of the Algorithms in the *Nine Chapters*)(1261)[14], and the *Yang Hui Suan Fa* (Yang Hui's Mathematical Methods) (1247—1275)[15], among other works. Yang achieved considerable success with his *duo ji shu* (methods for summing series), and improved the simple methods of calculation for multiplication, division, etc. The northern mathematical centre governed by the Jin and Yuan dynasties developed the *tian yuan shu* (celestial element method) for solving equations of higher degree, *er yuan shu* (two elements method), and *san yuan shu* (three elements method), for solving quadratic and cubic equations, the knowledge of *gou gu rong yuan* (measurements of right triangles and circles), and *duo ji shu* (methods for summing series). Li Ye (1192—1279) wrote the *Ce Yuan Hai Jing* (Sea Mirror of Circle Measurements)(1248)[16] and the *Yi Gu Yan Duan* (Old Mathematics in Expanded Sections) (1259)[17], both of which are devoted to the method of *tian yuan shu* (celestial element method). Wang Xun (1235—1281) and Guo Shoujing(1231—1316) used the *duo ji zhao cha* method (for summation of finite series) in the *Shou Shi Calendar*. Zhu Shijie, at the end of the 13th century, studied and accepted the best results and methods from the two mathematical centres in the south and the north, and as a result, his mathematical achievements were even greater than those of his predecessors.

2

Zhu Shijie, also know as Hangqing and Songting, resided in Yan (or

Yan Shan, today Beijing or its near vicinity). Neither the dates of his birth or death are known. Indeed, virtually no details of his personal life have survived, except for the fact that he was well-known as a great mathematician, having traveled throughout the country for more than 20 years. The number of people coming from all directions to see him increased daily. [18] His *Suan Xue Qi Meng* (Introduction to Mathematical Studies) [19] and the *Jade Mirror of the Four Unknowns* were both successively published at Yangzhou in 1299 and 1303, respectively. Zhu is one of the few professional mathematicians and mathematical educators in the history of China.

There is some introductory information at the beginning of the *Suan Xue Qi Meng*. Then the work is divided into three books which include 20 sections and 259 problems. Its contents concern many aspects of the mathematics of the Yuan dynasty, such as the operations and simple calculating methods of multiplication and division, root extraction methods, the *tian yuan shu* (celestial element method), finding solutions for simultaneous equations, methods for summing finite series, and so on. The contents range from the simple to the profound, and represent a more or less integrated system. It served no doubt as a teaching manual for Zhu Shijie to teach his students. The methods improving the rod arithmetic for multiplication and division were an important aspect of mathematics in the Song and Yuan dynasties. Because this sort of mathematics was a mundane activity related to commerce, some great scholarly mathematicians such as Li Ye disdained it. However, Yang Hui, Zhu Shijie and others regarded it as very important. Zhu Shijie evem turned all kinds of useful formulas for simple calculations of multiplication and division into rhymed verses that were easy to memorize and much easier to use than Yang Hui's. Some of these were especially suited for calculating with the abacus, and the develop-

ment of simpler calculating methods for multiplication and division helped bring the abacus into prominence. In the Ming dynasty, the abacus ultimately took the place of counting rods, and it remains a very useful tool in China, Japan, and Southeast Asia even today.

The *Jade Mirror of the Four Unknowns*, consisting of three books divided into 24 sections, contains 288 problems. The principles of both ancient and modern methods of elimination including four kinds of diagrams are included at the beginning of the work. And there are four introductory problems in the *si xiang xi cao jia ling zhi tu* (a chart of the detailed solutions for the four exemplary problems with one to four unknowns), which demonstrate the solutions for the "celestial element method" (tian yuan shu), the "two elements method" (er yuan shu), the "three elements method" (san yuan shu), and the"four elements method"(si yuan shu). The main achievements of the *Jade Mirror of Four Unknowns* are the"four elements method" for the solution of a group of simultaneous equations of higher degree in as many as four unknowns and the procedure for computing the sum of arithmetic series of higher degree and the method of higher power finite differences.

The *Si Ku Quan Shu* (an encyclopedic collection of classic texts) compiled during the Qian Long period, did not include the *Introduction to Mathematical Studies* or the *Jade Mirror of the Four Unknowns*, both of which had been lost. At the beginning of the Jia Qing period, Ruan Yuan (1764—1849) saw a copy of the *Jade Mirror of the Four Unknowns* in Zhejiang. Thereafter, among those who studied this text were Li Rui (1768—1817), Shen Qinpei [20], Luo Shilin (1789—1853)[21], Dai Xu (1805—1860)[22], Li Shanlan (1811—1882)[23], Ding Quzhong [24], Hua Hengfan (1833—1902)[25], and Zhou Da [26]. They did significant work for the method of elimination applied to the four elements and for the *duo ji zhao cha* method.

3

The numerical solution of equations of higher degree is one of the advanced branches of traditional Chinese mathematics. The *fang cheng* (rectangular arrays) method of ancient China is closely related to the modern theory of linear equations. The methods for solving equations with one unknown were all called *kai fang shu* (method of extracting roots). The *ping fang shu* (method of extracting square roots) and *li fang shu* (method of extracting cube roots) as applied in the *shao guang* (diminishing breadth) chapter of *The Nine Chapters of Mathematical Procedures* are the earliest methods for the extraction of roots in the history of Chinese mathematics. With the improvements introduced by Liu Hui, in the *Sun Zi Suan Jing*(Master Sun's Arithmetical Manual), and by others, increasingly accurate methods were used to solve quadratic and cubic equations down to the time of Zu Chongzhi [27] and Wang Xiaotong [28] (7th century). Zu Chongzhi could even solve equations with negative coefficients. Jia Xian extended the traditional methods to solve equations of higher degree, which was called *li cheng shi suo fa* (table of binomial coefficients and methods for equations of higher degree), in the course of which he introduced the Jia Xian Triangle as the *li cheng*(table). He also created the *zeng cheng kai fang fa*, a method by which the coefficients of the JiaXian Triangle were generated through successive additions and multiplicati-ons. [29] During the Song and Yuan periods, Qin Jiushao, Li Ye, Zhu Shijie and others took the *zeng cheng. kai fang fa*(the method of extracting roots by addition and multiplication) as a central result, and used the *zeng fu kai fang shu*(a method for extract-

ing roots with positive and negative coefficients) as a means of providing complete solutions in extracting positive roots of equations of higher degree. Zhu Shijie modified the *lian zhi tong ti shu* (previously used by Qin Jiushao and Li Ye), and called this the *zhi fen* method (for solving equations of the form $ax^n - a_0 = 0, a \neq 1$).

There was no uniform way of writing equations before the Song and Yuan dynasties. In order to lay out equations on the counting board, clever methods were usually needed with respect to different problems. Mathematicians created the *tian yuan shu* (celestial element method) in the Jin and Yuan periods. The so-called "celestial element method" is the method whereby solutions are obtained from equations of any degree in one unknown. This process is called the elimination of *ruji*.

As Zu Yi explained in the Preface to the *Jade Mirror*:"Later we find the element *tian* in the book *Yi Gu* written by Jiang Zhou of Pingyang, in the book *Zhao Dan* by Li Wenyi of Bolu, in the *Qian Jing* by Shi Xindao of Luquan, in the *Ru Ji Shi Suo* by Liu Ruxie of Pingshui, and in the work of Yuan Yu of Jiang." This roughly describes the progress of the origin, development, and eventual perfection of the *tian yuan shu*. Li Ye also employed a method using *tian* and *di* for the positive and negative powers of the element *tian*[30]. By canceling the element *di* which expressed negative powers, Le Ye only used the element *tian*. He ascertained its power according to its relative position with respect to *yuan* or *tai*, which expressed constant terms. Li Ye at first placed the positive powers at the top and the negative powers at the bottom of the counting board in his *Ce Yuan Hai Jing* (Sea Mirror of Circle Measurements, 1248, 1298 CE). He subsequently revised this in his *Yi Gu Yan Duan* (Old Mathematics in Expanded Sections, 1259), which then be-

came the standard format for the *tian yuan shu*. Zhu Shijie always used this layout in his *Suan Xue Qi Meng* (Introduction to Mathematical Studies) and *Si Yuan Yu Jian* (Jade Mirror of the Four Unknowns).

It was doubtless under the influence of the *fang cheng shu* (rectangular arrays method) in *The Nine Chapters of Mathematical Procedures* that the *tian yuan shu* was generalized to accommodate two, three, and four unknown elements. These were called the *er yuan shu* (two elements method), the *san yuan shu* (three elements method), and *si yuan shu* (four elements method) for solutions of equations of higher degree in two, three and four unknowns, respectively. When Zu Yi referred to this process in his preface to the *Jade Mirror*, he said:"It is from these books that Li Dezai of Pingyang obtained the materials for his book, *Liang Yi Qun Ying Ji Zhen* (A Collection of Precious Problems in Two Unknowns), which also contains the unknown element *di*. In the *Qian Kun Kuo Nang* (A Work on the Two Unknowns) by Liu Dajian Runfu, the brother of Xing Song (his another name is Xing Bugao) of Huoshan, there are two problems at the end of the book with the unknown element *ren*. My friend Zhu Hanqing of Yan Shan has for many years shown a deep interest in mathematics. He examined various studies of the *Nine Chapters*, especially with respect to the four elements *tian*, *di*, *ren* and *wu*." [31] This makes it clear that the creator of the four unknown elements method is Zhu Shijie, also known as Zhu Hanqing.

Muo Ruo said, the layout of four unknown elements on the counting board is as follows:"The method (of the four unknown elements) for solving equations puts the 元气 *yuan qi*(the zero or empty element)in the centre, the element 天 *tian* (heaven) at the bottom, the element 地 *di* (earth) on the left, the element 人 *ren* (man) on the right, and the element 物 *wu* (thing)

at the top." [32] The *er yuan shu* (method of two unknown elements) involves two equations, each of which has two unknown elements; the *san yuan shu* (method of three unknown elements) involves three equations, each of which has three unknowns; and the *si yuan shu* (method of four unknown elements) involves four equations, each of which has four unknown elements. If x represents the unknown element *tian*, y the unknown element *di*, z the unknown element *ren*, and u the unknown element *wu*, then the layout for the method of four unknown elements is as follows:

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots y^3u^3 & y^2u^3 & yu^3 & u^3 & u^3z & u^3z^2 & u^3z^3 \cdots \\
\cdots y^3u^2 & y^2u^2 & yu^2 & u^2 & u^2z & u^2z^2 & u^2z^3 \cdots \\
\cdots y^3u & y^2u & yu & u & uz & uz^2 & uz^3 \cdots \\
\cdots y^3 & y^2 & y & \text{太} & z & z^2 & z^3 \cdots \\
\cdots xy^3 & xy^2 & xy & x & zx & z^2x & z^3x \cdots \\
\cdots x^2y^3 & x^2y^2 & x^2y & x^2 & zx^2 & z^2x^2 & z^3x^2 \cdots \\
\cdots x^3y^3 & x^3y^2 & x^3y & x^3 & zx^3 & z^2x^3 & z^3x^3 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

The product of both powers of two elements that one is border upon the other is placed at the cross-point of corresponding rows on the horizon and on the verticality. The product of both powers of two elements that one is not border upon the other is placed at their crevice.

The elimination method of the four unknown elements is namely an elimination method applied to equations of higher degree in more than two unknowns. It is the outstanding creation of the *Jade Mirror of the Four Unknowns*, that is, the four unknown elements are reduced to three elements, then to two elements, and finally to a single equation of higher degree in one

unknown. This can then be solved using the *zheng fu kai fang* method (in essence the same as the *zeng cheng kai fang* method, similar to Horner's method). However, concerning the actual process of this elimination procedure, in discussing the *si xiang xi cao jia ling zhi tu* (introductory problems), and specifically the third and fourth problems (*san cai yun yuan* — operations with three unknowns, and *si xiang hui yuan* — operations with four unknowns), Zhu Shijie only speaks of *ti er xiao zhi* (eliminating by reducing the degree), *ren yi tian wei* (exchanging the positions of tian and ren), and "wu yi tian wei" (exchanging the positions of *tian* and *wu*). And after reducing equations with three or four unknown elements (*san cai yun yuan* or *si xiang hui yuan*) to two expressions each of which has only two unknown elements, in reference to the second, third and fourth problems he only said *hu yin tong fen xiang xiao* (equalizing coefficients for elimination). All of his explanations are very brief. Zu Yi concluded, "By moving expressions from top to bottom, from right to left, by applying multiplication and division, by various methods of rearranging the terms, by assuming the unreal for the real, by using the imaginary for the true, by using the positive and negative, by keeping some and eliminating others, and then changing the position of the counting rods and by attacking (the problem) from the front or from one side, as shown in the four examples, he finally works out the process of elimination in a profound yet natural manner." [33] Nevertheless, the main points of this procedure remained obscure and were not easy to understand. Among those who have studied the method of elimination of the four unknown elements, it is commonly said that Shen Qinpei's understanding as reflected in his unpublished mid-Qing dynasty manuscript, *Si Yuan Yu Jian Xi Cao* (Detailed Explanation of the Jade Mirror, ca. 1829) accords quite closely with the original

meaning of the *Jade Mirror*.

The method for eliminating three and four unknown elements is called *ti er xiao zhi*. This means that expressions in four or three unknowns are reduced to expressions in two unknowns. For example, to eliminate the element *ti* (or y) from equations in three unknown elements, consider the following:

$$A_2y^2 + A_1y + A_0 = 0$$

$$B_2y^2 + B_1y + B_0 = 0 \qquad (1)$$

Where all of the coefficients A_2, A_1, A_0; B_2, B_1, B_0 are polynomials in x and z alone. The above two equations can then be rewritten as follows, the rearrangement of terms meant to eliminate the element y^2:

$$(A_2y + A_1)y + A_0 = 0$$

$$(B_2y + B_1)y + B_0 = 0$$

It is then a simple matter of cross-multiplying the first equation by B_0 and the second by A_0, and subtracting the two expressions which gives:

$$(A_2B_0 - A_0B_2)y + (A_1B_0 - A_0B_1) = 0 \qquad (2)$$

All the same, it is then a simple matter of cross-multiplying the first equation by B_2 and the second by A_2, and subtracting the two expressions, which gives:

$$(A_1B_2 - A_2B_1)y + (A_0B_2 - A_2B_0) = 0 \qquad (3)$$

Taking the expressions (2) and (3) and one of the two expressions in (1) above as simultaneous equations, the same process may be repeated to yield the following expression:

$$C_1y + C_0 = 0 \qquad (4)$$

As before, C_1 and C_0 are polynomials in x and z alone. Using the same procedure as outlined above, equations which include only x and z, but not y,

can thus be found.

After carrying out the above *ti er xiao zhi* elimintion process in the method of four unknown elements, the positions of *tian* and *ren* are exchanged on the counting board. If the element *di* (y) is eliminated from an expression of three elements, the remaining two expressions with the elements *tian* (x) and *ren* (z) need to be rotated 90 degrees (x' s thus move to the former y' s positions, and z's move to the former x's positions); thus elements in the fourth quadrant are moved to the third quadrant. Similarly, the positions of *tian* (x) and *wu* (u) may be exchanged as follows: if the element *tian* (x) is eliminated from an equation of four elements, the remaining equations with the elements *di*, *ren*, and *wu* need to be rotated on the counting board 180 degrees, taking the elements *wu* (u's) to where the *tians* (x's) were previously; this in effect rotates all of the elements from the first and second quadrants to the third and fourth quadrants. The equations in question remain invariant during this kind of transformation. It is because the third quadrant is often used in rod counting that this kind of transformation is done.

The *hu yin tong fen* (equalizing coefficients) method is used to reduce expressions in two unknowns whose coefficients are laid out in more than two rows on the counting board to expressions whose coefficients only occupy two rows on the counting board. Take for example the equations of nth degree in the two unknowns x and y:

$$A_n x^n + \cdots + A_2 x^2 + A_1 x + A_0 = 0$$

$$B_n x^n + \cdots + B_2 x^2 + B_1 x + B_0 = 0$$

Here the terms $A_n, \cdots, A_2, A_1, A_0, B_n, \cdots, B_2, B_1, B_0$ are all polynomials in which y occurs as the only unknown element; there is no element x in these expressions. Through successive applications of mutual multiplication and

cancellation, each of the terms $x_n, x_{n-1}, \cdots, x_2$ can be eliminated, finally reducing these expressions to two equations in two rows. [34]

Zhu Shijie's method of elimination continues to play a major role in contemporary research, even today. Wu Wenjun has said that he was inspired in his own work by the *fang cheng* method in *The Nine Chapters of Mathematical Procedures*, and Zhu Shijie's method of elimination of the four unknown elements. Combining these with other theories of modern mathematics, Wu Wenjun found the *san jiao hua zheng xu* method (the Well-Ordering Principle of Polynomial System triangulation), and "deduced solutions for completely solving simultaneous equations of higher degree according to this method." [35]

Among the problems solved by Zhu Shijie using the *tian yuan shu* (celestial element method), the *er yuan shu* (two elements method), the *san yuan shu* (three elements method), and the *si yuan shu* (four elements method), he often used the *duo ji shu* (the method of summing arithmetic series) and the *zhao cha shu* (the method of differences) to advance various hypotheses. Therefore, the *duo ji* and *zhao cha* methods represent other important achievements found in the *Jade Mirror of the Four Unknowns*.

There are summation formulas for arithmetic progressions in both the *Nine Chapters*, and in the *Zhang Qiujian Suan Jing* (Zhang Qiujian's Mathematical Manual), which is the more complete of the two. Agriculture and handicraft industries were developed in the Song and Yuan dynasties, and a considerable amount of grain, fruit, vegetables, pots, bottles, tiles, etc. were produced. Their quantities needed to be calculated, and the *duo ji* method emerged as a very useful one as the times required.

Shen Kuo (1031—1095) was a mathematician in the Northern Song dynasty. After studying problems of the summation of finite series, he ob-

served that a pile in the form of a *chu tong* (frustum of a rectangular-based pyramid) could not be analyzed in terms of the *chu tong* formula found in the *Nine Chapters* because *ji er you xi* (literally "the pile has spaces," meaning that computing the volume of a geometric solid like the *chu tong* is not the same as counting a number of discrete objects piled up in the shape of a *chu tong*, the former being a continuous body, the latter a compilation of discrete units piled one atop the other). For the latter pile of discrete objects stacked up in the shape of a *chu tong*, Shen Kuo created the *xiji* method (a method for summing finite series).[36] This method actually gives a summation formula for arithmetic series of the second order. During the Southern Song dynasty, Yang Hui brought forward several other *duo ji* formulas which also concerned problems of summing arithmetic series of the second order. Wang Xun, Guo Shoujing, and others solved computational problems involving the distances the sun and moon traversed daily in degrees in the *Shou Shi Calendar* using the *duo ji zhao cha* method. Moreover, Zhu Shijie raised the *duo ji zhao cha* method to a more advanced level.

Zhu Shijie advanced a number of summation formulas for a *san jiao duo* (lit."triangular pile"), a stack of objects with a triangular base. [37] Its formula for the general term is the sum of the first n terms of Zhu Shijie' s formula for a *jiao cao duo* (pile of hay, but this also serves as a technical mathematical term for the sum of the first n integers):

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{1}{2!} n (n + 1) \qquad \textit{jiao cao duo}$$

The summation of a *san jiao duo* (triangular pile) or *luo yi xing duo* (lit."the next level after *jiao cao duo*"; again, here this is a technical mathematical expression meaning that the sum of the *san jiao duo* series involves one iterated term more than that for the one just before it, namely the sum for the *jiao cao*

duo series) is:

$$S_n = \sum_{i=1}^{n} \frac{1}{2!} i(i+1) = \frac{1}{3!} n(n+1)(n+2) \qquad \textit{san jiao duo}$$

This is actually the problem of summing an arithmetic series of the second order, which was advanced by Yang Hui in the *bie nao* problem in the chapter on *shang gong* (construction consultations) in his *Xiang Jie Jiu Zhang Suan Fa* (Detailed Analysis of the Mathematical Rules in the *Nine Chapters*). Zhu Shijie also used *sa xing xing duo* (or *san jiao luo yi xing duo* (lit. "the next level after *san jiao duo*"), for which the formula is:

$$S_n = \sum_{i=1}^{n} \frac{1}{3!} i(i+1)(i+2) = \frac{1}{4!} n(n+1)(n+2)(n+3)$$

sa xing xing duo

The *san jiao sa xing xing duo* (or *sa xing geng luo yi xing duo*):

$$S_n = \sum_{i=1}^{n} \frac{1}{4!} i(i+1)(i+2)(i+3)$$
$$= \frac{1}{5!} n(n+1)(n+2)(n+3)(n+4)$$

The *san jiao sa xing geng luo yi xing duo*:

$$S_n = \sum_{i=1}^{n} \frac{1}{5!} i(i+1)(i+2)(i+3)(i+4)$$
$$= \frac{1}{6!} n(n+1)(n+2)(n+3)(n+4)(n+5)$$

Among these formulas, the sum of the first n terms of the former is exactly the nth term of the latter formula. Hence, the latter one is called the *luo yi xing duo* (one rank after the former). At the same time, all the terms of every pile are numbers that can be read off directly from the second, third, fourth, fifth, and sixth diagonals of the Jia Xian Triangle. And the sum of every pile is the nth number of the third, fourth, fifth, sixth, and seventh diagonals, respectively. This is the reason why Zhu Shijie used two groups of lines which run parallel to the left and right sides of the Jia Xian Triangle, connecting all of the numbers in each sequence. Although the original text of the *Jade Mirror* does not treat these series as systematically as the above

notation suggests, in fact, Zhu Shijie must have had some such complete system in mind, for he was even able to derive the summation formula for the first n terms of a *san jiao duo* (triangular pile) whose general term is given by the formula: $\frac{1}{p!} i (i + 1) (i + 2) \cdots (i + p - 1)$. The summation formula is then as follows:

$$S_n = \sum_{i=1}^{n} \frac{1}{p!} i (i + 1) (i + 2) \cdots (i + p - 1)$$
$$= \frac{1}{(p + 1)!} n (n + 1) (n + 2) \cdots (n + p).$$

Clearly, when $p = 1, 2, 3, 4, 5$, this gives the number of objects in every previous pile.

Zhu Shijie also used the summation formula of *lan feng xing duo*. Multiplying every term of the *san jiao duo* by the order of the term gives the *lan feng xing duo* ' s formula for the general term, namely: $\frac{1}{p!} i (i + 1) (i + 2) \cdots (i + p - 1) i$. The summation formula of all the former terms is then:

$$S_n = \sum_{i=1}^{n} \frac{1}{p!} i (i + 1)(i + 2) \cdots (i + p - 1) i = \frac{1}{(p + 2)!} n (n + 1)(n + 2) \cdots (n + p)[(p + 1) n + 1] .$$

When $p = 1, 2, 3, \ldots$, it is *si jiao duo*, *lan feng xing duo*, *san jiao lan feng xing duo*, or *lan feng geng luo yi xing duo*, and so on.

Moreover, Zhu Shijie also solved problems concerning more complicated *duo ji* (piles), such as *san jiao tai duo*, *si jiao tai duo*, etc.

In the section called "Ru Xiang Zhao Shu" (Men Summoned as Needed), Zhu Shijie made use of *zhao cha* (difference) formulas like the following:

$$f (n) = n \Delta_1 + \frac{1}{2!} n (n + 1) \Delta_2 + \frac{1}{3!} n (n - 1) (n - 2) \Delta_3 + \frac{1}{4!} n (n - 1) (n - 2) (n - 3) \Delta_4 .$$

In the above formula, Δ_1 is the first difference, Δ_2 is the second difference, Δ_3 is the third difference, and Δ_4 is the fourth difference. The coefficient of the second difference is the volume of the *jiao cao duo* (triangular haystack) with $n - 1$ as its base. The coefficient of the third difference is the volume of the *san jiao duo* (triangular pile) with $n - 2$ as its base. The coefficient of the fourth difference is the volume of a *sa xing xing duo* with $n - 3$ as its base. Zhu Shijie' s formula is exactly the same as the modern general formula. [38] It is obvious that *zhao cha* method reduces *duo ji* to the summation of every *san jiao duo* by the operation for differences. And every difference is uniquely determined by given *duo ji*. [39]

Most of these achievements of Zhu Shijie are well in advance of results achieved elsewhere only centuries later by European mathematicians, for example in the 17th, 18th and 19th centuries.

After the *Jade Mirror of the Four Unknowns*, Chinese mathematics suffered a disastrous decline. Regrettably, Qing mathematicians did not understand many of the significant mathematical contributions of the Song and Yuan dynasties, such as *zeng cheng kai fang fa* (method of extraction of roots by addition and multiplication), *tian yuan shu* (celestial element method), or the *si yuan shu* (method of the four unknowns). No one was interested in studying the important mathematical works of the Han, Wei, Southern and Northern dynasties, or of the Song and Yuan periods. Some works were even lost. Islamic mathematics and Western mathematics went on to surpass that of China, and China lost its status as a leading country in mathematics. But in retrospect, the *Jade Mirror of the Four Unknowns* may be regarded as the last great creative work produced during the most flourishing period of traditional Chinese mathematics, and is the most advanced of them all.

4

The Chinese text of the *Jade Mirror of the Four Unknowns* reproduced here is taken from a copy collated by Ding Quzhong in 1876, which was based on another by Luo Shilin[40]. Guo Shuchun translated the ancient text into modern Chinese, and provides some helpful collations and comments.

For the English text, we rely primarily on the translation by Ch'en Tsai Hsin (1879—1945) in the Library of the Institute for the History of Natural Science of the Chinese Academy of Sciences. Ch'en Tsai Hsin (also known as Ch'en Huamin), was born in Nanxin village (near today's Fragrant Mountain in the Haidian district of Beijing), Wanping county. As a child he studied at a missionary school, then went to Peking University (*Huiwen Daxue* was named Peking University in English before it was merged into Yenching University), from which he graduated in 1901. After teaching at the university, where he taught mathematics, he subsequently went to the United States where he studied at Columbia University's Teachers College. He obtained his master's degree from Columbia in 1912. It is known that he was awarded the Ph.D. sometime prior to 1913. Consequently, he may well be the first Chinese to have obtained a Ph.D. in mathematics. After returning to China from America, he continued to teach at Peking University. In 1919, Peking University was merged into Yenching University, which was established that same year. Ch'en Tsai Hsin was invited by the president of Yenching University, John Leighton Stuart (1876 — 1962), to serve as Chairman of the Department of Mathematics at the new university, a position Ch'en held until he retired from teaching due to a serious illness in 1936. In addition to mathematics, Ch'en Tsai Hsin was also interested in teaching

and studying history of mathematics, and during the period he was chairman of the department, he taught two courses on the history of mathematics. One was the history of mathematics in general; the other was devoted specifically to the history of Chinese mathematics, which is unusual even in present departments of mathematics in China.

While studying at Teachers College, Columbia University, Ch' en Tsai Hsin was encouraged by Dr. David Eugene Smith, an historian of mathematics, to undertake the English translation of the *Jade Mirror of the Four Unknowns*, which Ch'en began to do. With help from Ch'en's colleague, Emma Louise Konantz, in the Department of Mathematics at Yenching University, he eventually finished his translation sometime near the end of 1925. The results of their collaboration were sent to the Harvard-Yenching Institute in 1929, which in turn gave it to George Sarton (1884 — 1956), the famous historian of science, for his advice about its publication. Sarton's evaluation was very favorable, and he recommended that it should be published with some changes chiefly with respect to philological concerns. It is not known why it was not subsequently published. Sarton examined the manuscript again in 1940, and hoped that it would be possible to publish both the Chinese text and Ch'en's English translation in the Harvard-Yenching series. [41] This too never happened, possibly due to the disruption in academic life caused by World War II. Nevertheless, based upon Ch'en Tsai Hsin's translation and explanatory notes, it is clear that he had a deep appreciation for the elimination method of the four unknowns and the solutions this method made possible for such problems as those involving the *lian zhi tong ti* method (an extraction method for equations of the form $ax^n - a_0 = 0$, for which $a \neq 1$). His results are still valuable for current research on the history of Chinese

mathematics and for our understanding of traditional Chinese mathematics.

Guo Jinhai has revised the English manuscript. Because some pages are now missing, the translation of some problems cannot be found. These are problems 14-20 in the section on *Ming Ji Yan Duan* (Problems on Areas), and problem 13 in the section on *Bo Huan Jie Tian* (Land Measurements). Guo Jinhai has supplied their English translations. He also translated the supplemental commentaries of Guo Shuchun, and together they wrote this introduction in English. In addition to Ch'en Tsai Hsin's English preface and introduction which he translated into Chinese, Guo Jinhai also investigated related materials about the life and work of Ch'en Tsai Hsin, and was responsible for comparison for both the Chinese and English parts of the book. In what follows, Ch' en Tsai Hsin' s commentaries are labeled with a C or the Chinese character "陈"; G or the Chinese character "郭" denotes Guo Shuchun's supplementary comments. Moreover, Ch' en Tsai Hsin commonly placed the character *tai* beside an equation's constant term, but this is a mistake that goes back to the middle of the Qing dynasty. As a matter of fact, placing *tai* next to the constant term, or *yuan* beside the term of the first power, expresses a polynomial, but does not express a *kai fang shi* (what is now called an "equation in one unknown"). In the *tian yuan* method, once two equivalent *tian yuan* equations are reduced to the *kai fang* layout on the counting board by *ru ji xiang xiao* (a technical term for equalizing and then eliminating coefficients), the characters *tai* or *yuan* will be eliminated. [42] Therefore, we have omitted the characters *tai* and *yuan* from the commentaries by Ch'en Tsai Hsin. As for transliterations of Chinese words, Ch'en Tsai Hsin always used the Wade-Giles system. Instead, we have preferred to use modern Chinese *pinyin* spellings.

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【Notes】

[1] Zhu Shijie. *Jade Mirror of the Four Unknowns*. In Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993. Book 1, pp. 1205-1275. The work' s title was translated by Ch' en Tsai Hsin as *Precious Mirror of the Four Elements*. The same translation was used in *Chinese Mathematics. A Concise History* which is an English edition of *Zhong Guo Shu Xue Jian Shi* written by Li Yan and Du Shiran. (See Li Yan and Du Shiran. *Chinese Mathematics. A Concise History*. John N. Crossley and Anthony W.-C. Lun, trans. Oxford: Clarendon Press, 1987, p. 111); the work' s title was translated as *Jade Mirror of the Four Origins* in the English edition of Jean-Claude Martzloff' s *Histoire des mathêm atiques chinoises* (See Jean-Claude

Martzloff. *A History of Chinese Mathematics*. Stephen S. Wilson, trans. Berlin: Springer-Verlag, 1997, p. 17). Martzloff explains that the true significance of the title is best rendered as "Mirror [trustworthy as] jade [relative to the] four origins [unknowns]." See Martzloff 1997, p. 153. Given the Chinese title, 玉 *yu* means "jade" and 元 *yuan* means "element," which in the context of this work means "unknown," so we have translated Zhu Shijie's title as *Jade Mirror of the Four Unknowns* according to the suggestion given by Joseph W. Dauben. Note that *yuan* refers to "unknown" in the mathematical sense, and sometimes is translated here as such, or as "unknown element." For a detailed study of Zhu Shijie and his mathematics, consult Qian Baocong, ed. *Zhong Guo Shu Xue Shi* (History of Chinese Mathematics). Beijing: Science Press, 1964, pp. 175-205; Guo Shuchun and Liu Dun, ed. *Li Yan Qian Baocong kexueshi quanji*, Shenyang: Liaoning Education Press, 1998. Book 5, pp. 197-227; Du Shiran. *Zhu Shijie yanjiu* (Research on Zhu Shijie). In Qian Baocong, *et al.*, *Song Yuan shuxue shi lunwen ji*. Beijing: Science Press, 1966, pp. 166-209; John Hoe. *Les systèmes d'équations polynômes dans le Siyuan Yujian* (*1303*), Paris: Collège de France, Institut des Hautes Études Chinoises (Mémoires de l'institut des Hautes Études Chinoises, vol. 6); Andrea Bréard. *Re-kreation eines mathematischen Konzeptes im cinesischen Diskurs. "Reihen" vom 1. bis zum 19. Jahrhundert.* (*Boethius*, vol. 42). Stuttgart: Franz Steiner Verlag, 1999), pp. 178-264.

[2] *The Suan Shu Shu*, unearthed from a Han dynasty tomb at Zhangjiashan, Jiangling county in Hubei province in the 1980s, is a mathematical treatise written on bamboo slips. For a transcription and facsimile of the bamboo strips that comprise this document, see *Bamboo Slips of a Han Dynasty Tomb at Zhangjiashan* (*Tomb 247*). Beijing: Cultural Relics Publishing House, 2002; and Guo Shuchun. "Collation of the *Suanshushu*," *Zhongguo keji shiliao* (China Historical Materials of Science and Technology), 22 (3) (2001), pp. 202-219.

[3] *Zhou Bi Suan Jing*, annotated by Zhao Shuang, collated by Guo Shuchun and

Liu Dun, in *Suan Jing Shi Shu* (Ten Mathematical Classics). Guo Shuchun and Liu Dun, eds. Shenyang: Liaoning Education Press, 1998; Taipei: Jiu Zhang Press, 2001, pp. 29-80. See also the English translation: *Astronomy and Mathematics in Ancient China, the Zhou Bi Suan Jing*. Christopher Cullen, trans. Cambridge: Cambridge University Press, 1996.

[4] *Jiu Zhang Suan Shu*, annotated by Liu Hui and Li Chunfeng, collated by Guo Shuchun. Shenyang: Liaoning Education Press, 1990; the supplemented edition was publised by Liaoning Education Press and Jiu Zhang Press in 2004; also collated by Guo Shuchun, in *Suan Jing Shi Shu* (Ten Mathematical Classics), Guo Shuchun and Liu Dun, eds. Shenyang: Liaoning Education Press, 1998; Taipei: Jiu Zhang Press, 2001, pp. 81-244. See as well the English translation, Shen Kangshen, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, John Crossley and Anthony W.-C. Lun, trans. Oxford: Oxford University Press and Beijing: Science Press, 1999. See Chinese-French Edition: Karine Chemla and Guo Shuchun. *Les Neuf Chapitres: le classique mathèmatique de la Chine Ancienne*. Paris: Dunod Editeur, 2004.

[5] Liu Hui's theorem states: a cube is cut into two wedges. One of the wedges is called a 堑堵 *qiandu*. A qiandu is cut slantwise into one 阳马 *yangma*(a right quadrilateral pyramid) and one 鳖臑 *bienao* (a tetrahedron with four right-triangle sides). The proportion of the volume of a *yangma* to the volume of the *bienao* is two to one. This is the foundation for Liu Hui's theories for determining the volume of a polyhedron.

[6] David Hilbert."Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Kongre β zu Paris 1900," *Göttinger Nachrichten* (1900), pp. 253-297; English translation by Mary F. Winston,"Mathematical Problems: Lecture Delivered Before the International Congress of Mathematicians at Paris in 1900," *Bulletin of the American Mathematical Society*, 8 (1902), pp. 437-479; repr. in David Hilbert, *Gesammelte Abhandlungen*. Berlin: Springer-Verlag, 1935, Vol. 3, pp. 290-329; and in *Mathematical Developments Arising from Hilbert Problems*, Felix Brouder, ed., Providence,

RI: American Mathematical Society, 1976. Also translated into Chinese by Li Wenlin and Yuan Xiangdong, *Shuxue shi yiwen ji*. Shanghai: Shanghai Science and Technology Press, 1981, pp. 60-84.

[7]Liu Hui. *Sea Island Mathematical Manual*. Annotated by Li Chunfeng, collated by Guo Shuchun and Liu Dun, in *Suan Jing Shi Shu* (Ten Mathematical Classics), Guo Shuchun and Liu Dun, eds. Shenyang: Liaoning Education Press, 1998; Taipei: Jiu Zhang Press, 2001, pp. 245-255. See as well the English translation by Frank J. Swetz, *The Sea Island Mathematical Manual: Surveying and Mathematics in Ancient China*. University Park: Pennsylvania State University Press, 1992, and Shen Kangshen, John N. Crossley and Anthony W.-C. Lun, *The Nine Chapters on the Mathematical Art: Companion and Commentary*. Oxford: Oxford University Press and Beijing: Science Press, 1999, pp. 518-559.

[8] Guo Shuchun. *Gudai shijie shuxue taidou Liu Hui* (Liu Hui, an Authoritative Mathematician from Antiquity). Jinan: Shandong Education Press, 1992); also reprinted Taipei: Ming Wen Shu Ju, 1995.

[9] Wei Zheng. et al. *Shi Shu · Lü Li Zhi Shang*. Beijing: Zhonghua Press, 1973, p. 388. This work was written by Li Chunfeng.

[10] *Sun Zi Suan Jing*, collated by Guo Shuchun, in *Suan Jing Shi Shu* (Ten Mathematical Classics), Guo Shuchun and Liu Dun, eds. Shenyang: Liaoning Education Press, 1998; Taipei: Jiu Zhang Press, 2001, pp. 257-291. See the English translation of the *Sun Zi Suan Jing* in Lam Lay Yong and Ang Tian-Se, *Fleeting Footsteps. Tracing the Conception of Arithmetic and Algebra in Ancient China*. Singapore: World Scientific, 1992; repr. 2004.

[11]Zhang Qiujian. *Zhang Qiujian Suan Jing*, annotated by Li Chunfeng, collated by Guo Shuchun, in *Suan Jing Shi Shu* (Ten Mathematical Classics), Guo Shuchun and Liu Dun, eds.Shenyang: Liaoning Education Press, 1998; Taipei: Jiu Zhang Press, 2001, pp. 293-348.

[12] Jia Xian. *Huang Di Jiu Zhang Suan Jing Xi Cao*. Nearly two-thirds of the contents of this work survive in Yang Hui' s *Xiang Jie Jiu Zhang Suan Fa*. See: Guo

Shuchun,"Huang Di Jiu Zhang Suan Jing Xi Cao Chu Tan"(An Elementary Exploration of the Huang Di Jiu Zhang Suan Jing Xi Cao), *Ziran kexue shi yanjiu* (Studies in the History of Natural Sciences), 7 (4) (1988), pp. 328-334.

[13] Qin Jiushao. *Shu Shu Jiu Zhang*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 439-724. See also the translation by Ulrich Libbrecht, *Chinese Mathematics in the Thirteenth Century: the Shu-shu chiu-chang of Ch 'in Chiu-shao*, Cambridge, MA: MIT Press, 1973.

[14] Yang Hui. *Xiang Jie Jiu Zhang Suan Fa*. In Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 949-1043.

[15] Yang Hui. *Yang Hui Suan Fa*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 1047-1117. The entire work has been translated into English by Lam Lay Yong, *A Critical Study of the Yang Hui Suan Fa. A Thirteenth-century Chinese Mathematical Treatise*. Singapore: Singapore University Press, 1977.

[16] Li Ye. *Ce Yuan Hai Jing*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 729-869.For a detailed study of Li Ye and his mathematics, consult Qian Baocong, ed. *Zhong Guo Shu Xue Shi* (History of Chinese Mathematics). Beijing: Science Press, 1964, pp. 168-178; Guo Shuchun and Liu Dun, ed. *Li Yan Qian Baocong kexueshi quanji*. Shenyang: Liaoning Education Press, 1998. Book 5, pp. 186-196; Mei Rongzhao. *Li Ye Ji Qi Shu Xue Zhu Zuo*. In Qian Baocong, *et al*., *Song Yuan shuxue shi lunwen ji*. Beijing: Science Press, 1966, pp. 104-148; Karine Chemla. *The Structures of Sea-mirror of Circle-measurements by Li Ye and His Expression of Mathematical Knowledge* (translated into Chinese by Guo Shirong). In Li Di, ed. *Shuxueshi yanjiu wenji*. Hohhot: Inner Mongolia University Press, and Taipei: Jiu Zhang

Press, 1993. Book 5, pp. 123-142; Karine Chemla. "Du parallélisme entre énoncés mathématiques, analyse d'un formulaire rédigé en Chine au 13 siècle," *Revue d'histoire des sciences*, 43 (1) (1990), pp. 57-80.

[17] Li Ye. *Yi Gu Yan Duan*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 873-941. For a study of this text, see Lam Lay Yong and Ang Tian-Se, Li Ye and his *Yi Gu Yan Duan* (Old Mathematics in Expanded Sections), *Archives for History of Exact Sciences*, 29(3) (1984), pp. 237-266.

[18] Mo Ruo. *The Former Introduction of Jade Mirror of the Four Unknowns*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, p. 1205.

[19] Zhu Shijie. *Suan Xue Qi Meng*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 1123-1200. Excerpts from this text are translated into German in Andrea Bréard. *Re-kreation eines mathematischen Konzeptes im cinesischen Diskurs. "Reihen" vom 1. bis zum 19. Jahrhundert.* (*Boethius*, Vol. 42). Stuttgart: Franz Steiner Verlag, 1999), pp. 378-387.

[20] Shen Qinpei. *Si Yuan Yu Jian Xi Cao*. For a transcription of this text, see Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 5, pp. 227-432.

[21] Luo Shilin. *Si Yuan Yu Jian Xi Cao*. Shanghai: Commercial Press, 1937.

[22] Dai Xu. *Si Yuan Yu Jian Xi Cao*. Manuscript. In the National Tsinghua University Library, Taiwan.

[23] Li Shanlan. *Duo Ji Bi Lei*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 5, pp. 911-963; and Li Shanlan. *Si Yuan Jie*. In Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui ·*

shuxue. Zhengzhou: Henan Education Press, 1993, Book 5, pp. 967-998.

[24] Ding Quzhong. *Si Xiang Jia Ling Xi Cao*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 1276-1280.

[25] Hua Hengfang. *Ji Jiao Shu*. See Hua Hengfang. *Xing Su Xuan Suan Gao*. Guang Xu Ren Wu Diao Liang Xi Hua Shi Cang Ban.

[26] Zhou Da. *Duo Ji Xin Yi*. Manuscript. In the Library of the Institute for the History of Natural Science of the Chinese Academy of Sciences, Beijing.

[27] Qian Baocong, ed. *Zhong Guo Shu Xue Shi* (History of Chinese Mathematics). Beijing: Science Press, 1964; and Guo Shuchun and Liu Dun, ed. *Li Yan Qian Baocong kexueshi quanji*. Shenyang: Liaoning Education Press, 1998. Book 5, pp. 97-98.

[28] Wang Xiaotong. *Ji Gu Suan Jing*. Collated by Guo Shuchun, in *Suan Jing Shi Shu* (Ten Mathematical Classics), Guo Shuchun and Liu Dun, eds. Taipei: Jiu Zhang Press, 2001, pp. 413-442.

[29] *Yong Le Da Dian Suan Shu*. See Guo Shuchun, ed. *Zhongguo kexue jishu dianji tonghui · shuxue*. Zhengzhou: Henan Education Press, 1993, Book 1, pp. 1416-1427.

[30] Li Ye. *Jing Zhai Gu Jin Tou*. Beijing: Zhonghua Press, 1995, p. 32.

[31] Zu Yi. The Latter Introduction of *Jade Mirror of the Four Unknowns*. See Guo Shuchun, ed. *Zhongguo Kexue jishu dianji tonghui·shuxue*. Zhengzhou:Henan Education Press, 1993, Book 1, p. 1206.

[32] Cf. [18]. This is all discussed at length, as is the diagram below, in Andrea Bréard. *Re-kreation eines mathematischen Konzeptes im cinesischen Diskurs. "Reihen" vom 1. bis zum 19. Jahrhundert*. (Boethius, Vol. 42). Stuttgart: Franz Steiner Verlag,

1999), pp. 178-188.

[33] Cf. [31] .

[34] Li Zhaohua."Si Yuan Xiao Fa De Zeng Gen Yu Jian gen WenTi," (On Problems of Extraneous and Lost Roots of Equations in the Method of the Four Elements), *Ziran kexue shi yanjiu* (Studies in the History of Natural Sciences), 21 (1) (2002), pp. 12-20.

[35] Wu Wenjun."Jie Fang Cheng Qi Huo SOLVER Ruan Jian Xi Tong Gai Shu," in *Shuxue de shijian yu renshi* (Mathematics in Theory and Practice), 2 (1986); and Wu Wenjun. *Wu Wenjun Lun shuxue ji xie hua* (Jinan: Shandong Education Press, 1995), pp. 491-505.

[36] Shen Kuo. *Meng Xi Bi Tan*. Collated by Hu Daojing. Shanghai: Shanghai Classics Publishing House, 1987, p. 574. For discussion of Shen Kuo's "*chutong with spaces*"("chutong mit Lücken") see Andrea Bréard. *Re-kreation eines mathematischen Konzeptes im cinesischen Diskurs."Reihen" vom 1. bis zum 19. Jahrhundert.* (*Boethius*, vol. 42). Stuttgart: Franz Steiner Verlag, 1999), pp. 110-118.

[37] For detailed discussion of this and the different finite series that Zhu Shijie considered in the *Jade Mirror*, see Qian Baocong, ed. *Zhong Guo Shu Xue Shi*(History of Chinese Mathematics). Beijing: Science Press, 1964, pp. 175-205; Guo Shuchun and Liu Dun, ed. *Li Yan Qian Baocong kexueshi quanji*. Shenyang: Liaoning Education Press, 1998. Book 5, pp. 197-227; Du Shiran. *Zhu Shijie yanjiu* (Research on Zhu Shijie). In Qian Baocong, *et al., Song Yuan shuxue shi lunwen ji*. Beijing: Science Press, 1966, pp. 166-209; the chapter devoted to Zhu Shijie in Andrea Bréard. *Re-kreation eines mathematischen Konzeptes im cinesischen Diskurs."Reihen" vom 1. bis zum 19. Jahrhundert*. Boethius, Vol. 42. Stuttgart: Franz Steiner Verlag, 1999, pp. 189-264.

[38] Du Shiran. *Zhu Shijie yanjiu* (Research on Zhu Shijie). In Qian Baocong, *et al., Song Yuan shuxue shi lunwen ji* (Beijing: Science Press, 1966), pp. 185-196.

[39] Li Zhaohua and Joseph C. Y. Chen."Zhu Shijie Zhao Cha Shu Tan Yuan," (A Further Investigation on Zhu Shijie's Interpolation Formula), *Ziran kexue shi yanjiu* (Studies in the History of Natural Sciences), 19 (1) (2000), pp. 30-39.

[40] Cf.[1].

[41] George Sarton. *Introduction to the History of Science*. Vol. III: *Science and Learning in the Fourteenth Century*. Baltimore: the Williams & Wilkins Company, 1953, p. 703.

[42] Guo Shuchun, Tian Miao, and Zou Dahai. *Cheng Ji Zhuo Zhu De Zhong Guo Shu Xue*. Shenyang: Liaoning Classics Publishing House, 1995, pp. 129-130.

序

陈在新

对数学史与东方文化感兴趣的人们都知道朱世杰。然而，他的名著《四元玉鉴》尚无完整的翻译。当我在哥伦比亚大学师范学院学习的时候，史密斯（David Eugene Smith）博士鼓励我承担这项翻译工作，那是我关于《四元玉鉴》翻译工作的开始。随着时间的推移，我已经认识到这部著作的重要性，且怀着其不仅可以填补思想发展史上的缺憾，而且可以增长对古代中国文化的了解的希望，高兴地将它介绍给大家。

我非常感谢我的合作者燕京大学数学教授寇恩慈（Emma Louise Konantz）的帮助。她不但将这部著作润色成目前的英语形式，而且提出了许多有价值的建议。

在可行之处，我们保持了中文符号。多数情况下，还是尽可能地使用相应的英语表述。

中国北平

1925年11月22日

PREFACE

by Ch'en Tsai Hsin

Those who have been particularly interested in the history of mathematics and in the Eastern culture have known the work of Zhu Shijie. However, no complete translation of his great work *Jade Mirror of the Four Unknowns* has ever been made. When studying in Teachers College, Columbia University I was encouraged by Dr. David Eugene Smith to undertake such translation and at that time made a beginning. As the years have passed I have realized the importance of this work and am now happy to present it to the public in the hope that it may not only fill a long-felt want in the history of the development of thought but it may also give an increased appreciation of the old Chinese culture.

I wish to express my great appreciation of the help given by my co-worker, Emma Louise Konantz, Professor of Mathematics in Yenching University, who has put this work into its present English form and has made many valuable suggestions.

We have maintained the Chinese notation wherever feasible. In most cases it has seemed best to use the English equivalents.

Peking, China

November 22, 1925

导 言

陈在新

【中译】

17世纪末期，一篇由一些罗马天主教传教士译自欧洲文献的代数论文，进呈给康熙皇帝。他将它交给当朝著名的算学家梅珏成。梅珏成对此书考察一番后断言，其原理与中国古代“天元一术”中所发现的极为相似。那是13世纪或更早之时中国的代数。他也认为西方的代数来自于东方，且Algebra一词的意思是“东来法”。我们在这里并非有意讨论这门学科的起源，而是将与朱世杰这部著作相关的中国代数学史做一个简短的交待。其译文将构成本书的主体部分。

朱世杰13世纪出生于燕山。那时中国有许多杰出的数学家，如，秦九韶、郭守敬、杨辉，以及李冶等。在祖颐季贤父的序中列出了一个名单，包括蒋周、李文一、石信道、刘汝谐，以及其他的一些人。他们这些人对于中国代数学的发展做出了重要的贡献。据我们所知，这些人的著作已经遗失，但从所保留下来的书名来看，这些是朱世杰为他的著作所获取的资料。

朱世杰是两部著作的作者。第一部为1299年出版的《算学启蒙》，第二部为四年后出版的《四元玉鉴》，即本书的主题。无疑，这些著作是为其从各地云集而来的学生撰写的。在第一部著作中，他论述了加法、减法、乘法、除法、测量法、测量平面与立体几何中不同图形的法则、盈不足术、解齐次方程(组)及开方法，也包括含有一个未知数的高次方程的解法。这部著作分为三卷，20门，259问。它在中国长时间失传，不过最后它在朝鲜被发现，且此后在中国几次重印。

第二部著作所论述的全为高次方程。使用的方法将在注解中阐明。不过，它与现代方法并无本质的区别。

INTRODUCTION

by Ch'en Tsai Hsin

【Original Text】

Toward the end of the 17th century, a treatise on algebra, translated from European sources by some of the Roman Catholic missionaries, was laid before Emperor Kang Xi, who placed the book in the hand of Mei Juecheng, a well known mathematician of the empire. The latter examined the book and asserted that the principle was exactly the same as that found in the old Chinese *Tian Yuan Yi Shu* (method by using the element *tian*), a Chinese algebra known in the 13th century or even earlier. He claimed also that the Western world derived this science from the East and the word algebra meant"came from the East." It is not my intention to enter into the origin of the science, but rather to give a short account of the history of Chinese algebra in connection with the work of Zhu Shijie, a translation of which will form the major part of this work.

Zhu was born at Yanshan in the 13th century. At that time there were living in China many prominent mathematicians such as Qin Jiushao, Guo Shoujing, Yang Hui, and Li Ye. A list of names of like note appears in the introduction of this book by Zu Yi Ji Xian Fu, where mention is made of men like Jiang Zhou of Pingyang, Li Wenyi, Shi Xindao, Liu Ruxie, and many others, who contributed much to the science of mathematics by the development of algebra. Although the books of these men are lost we learn, nevertheless, from the titles which still remain that they were the materials which Zhu Shijie obtained for his work.

Zhu was the author of two books, the first was *Suan Xue Qi Meng* (Introduction to Mathematical Studies), published in 1299, and the second, which appeared four years later, was *Si Yuan Yu Jian*, the subject of the present work. These books were written without any doubt for the use of his pupils who came from various places. In his first book, he treats of addition, subtraction, multiplication, division, mensuration, rules for measuring various geometric forms, both plane and solid , excess and deficiency, the solution of simultaneous equations and evolution, which means also the solution of higher equations of one unknown quantity. The book was divided into three parts, containing 20 sections with 259 problems. It disappeared for a long period from China but was finally discovered in Korea and has since been reprinted several times in China.

The second book treats entirely of equations of higher degree. The method employed will be set forth in the notes. It may be said, however, that it does not differ materially from modern method.

《四元玉鉴》前序

[元] 莫 若

【原文】

数，一而已。一者，万物之所从始。故《易》："一，太极也。"一而二，二而四，四而八，生生不穷者，岂非自然而然之数邪？河洛图书[1]泄其秘，《黄帝九章》[2]著之书，其章有九，而其术则二百四十有六，始方田，终勾股，包括三才，旁通万有，凡言数者，皆莫得而逃焉。如《易》之大衍[3]，《书》之秝象[4]，《诗》之万亿及秭[5]，《礼记》之三千三百，《周官》之三百六十，数之见于经者，盖不特《黄帝九章》为然也。

自后世明算之科不设，而此学寖失其传。由是秝法之进退畸盈，农田之方圆曲直，以至斗、升、勺、合[6]，豪、厘、丝、忽[7]，往往皆不能尽其法者，又岂非古学之无传，而学者莫知所依据邪？

燕山松庭朱先生[8]以数学名家周游湖海二十余年矣，四方之来学者日众。先生遂发明《九章》之妙，以淑后学。为书三卷，分门二十有四，立问二百八十有八，名曰《四元玉鉴》。

其法以元气居中，立天元一于下，地元一于左，人元一于右，物元一于上。阴阳升降，进退左右，互通变化，错综无穷。其于盈朒隐互、正负方程、演段开方之术，精妙玄[9]绝。其学能发先贤未尽之旨，会万理而朝元，统三才而归极，乘除加减，钩深致远，自成一家之书也。

方今尊崇算学，科目渐兴，先生是书行将大用于世。有能执此以往，则古人格物致知之学，治国平天下之道，其在是矣。有志于学者，可不服膺此书云？

大德癸卯[10]上元日临川前进士莫若序

【注释】

[1]这是一个古老的神话传说：伏羲氏(公元前2852年)看到一匹龙马背负“河图”从黄河出；同样传说夏朝的第一位君主大禹看到有神龟背负“洛书”从洛水出。(陈)

[2]黄帝是公元前26世纪的一位君主。他曾命其大臣隶首撰写《九章算术》。(陈)

[3]大衍是求解不定问题的一种方法。它的一般法则由秦九韶在《数书九章》中给出。(陈)

[4]秝象，关于天文方面的计算，如历法的推算。秝，即历。(陈)

[5]万亿及秭，即10,000、100,000,000,以及1,000,000,000,000。《诗经》曾有所记载。(陈)

[6]合为一个容量单位。见文末附录朱世杰在《算学启蒙》中使用的度量单位换算表。(陈)

[7]忽为一个长度单位。见文末附录朱世杰在《算学启蒙》中使用的度量单位换算表。(陈)

[8]朱松庭是朱世杰的号。(陈)

[9]玄，原刻本避康熙名讳而改作“元”。下凡被改为“元”者，径改回“玄”，不再注。(郭)

[10]癸卯即1303年。(郭)

【今译】

数，只不过是一罢了。一是万物所从之开始的数。所以《周易》说：“一就是太极。”一衍生出两仪，两仪衍生出四象，四象衍生出八卦，生生不息，衍变是没有穷尽的，这难道不就是自然而然的数吗？河图、洛书泄露了数学的奥秘，《九章算术》将它们写到书中。它有九章，246个

题目，从方田开始，以勾股终结，包括了天、地、人间万物的数学问题。凡是谈到数学问题的，都不能逃出它的范围。另外，如《周易》有大衍之数，《尚书》有历法天象问题，《诗经》有万、亿、秭等大数，《礼记》有三千、三百，《周官》有三百六十，都是数学问题见于经典的例子，不特《九章算术》是这样的。

自从后来不再设立明算科，这种学问渐渐失传。因此，历法的进退、奇零、盈缩，农田的方圆、曲直，以至于斗、升、勺、合与毫、厘、丝、忽等度量衡，往往都不能符合法度，又难道不是因为古代的数学没有传下来，而使后之学者不知所据吗？

燕山朱松庭先生以数学名家周游湖海20余年。四面八方来向他学习的人日益增多。先生于是阐发《九章算术》的奥妙，以教导后学。撰成一部书，凡三卷，24门，设立288道问题，书名叫做《四元玉鉴》。

它的方法是将元气放置在中央，立天元一在下方，立地元一在左方，立人元一在右方，立物元一在上方。各元能够阴升阳降，左进右退，使之互相通达变化，错综无穷。他对于盈不足术、正负术、方程术、演段术和开方术等的运用也是精妙绝伦。他的学说能阐发先贤发挥不充分的要旨，能会通万物之理而可以通达数学之精神，能统摄天、地、人三才的数学问题而登峰造极，运用加减乘除，钩深致远，是自成一家的著作。

现今推崇数学，各种科目日渐兴盛，先生的这部书行将大有用于世事。如果有人能掌握这部书走向社会，那么古人格物致知的学问，治国平天下的道理，都在这里了。有志于学问的人，难道可以不服膺这部书吗？

大德癸卯年正月十五日临川前进士莫若序

THE FORMER INTRODUCTION OF *JADE MIRROR OF THE FOUR UNKNOWNS*

by Mo Ruo

Number is unity for unity is the beginning of all things. In the book of *Yi* (changes) unity is the *tai ji* (the great extremity). One produces two; two produces four; four produces eight; and so on to infinity. Is this not the natural way of number? The "Tu Shu [1]" from He Luo betrays the secret and the *Nine Chapters* of Huang Di [2] expresses it in words. These nine chapters beginning with *fang tian* (mensuration) and ending with *gou gu* (right triangles) containing 246 rules, including *san cai* (three talents or elements) and lead to the whole universe, hence those who speak of number are unable to reach its limit. *Dayan*[3] in the book of *Yi*, *li xiang*[4] in the book of *Shu*, *wan*, *yi* and *zi*[5] in the book of *Shi*, *san qian* and *san bai* in the book of *Li*, and 360 in the book of *Zhou Guan*, all these inform us that numbers are used in the different classics as well as in the *Nine Chapters*.

For a long period no place was found for the teaching of mathematics, therefore the science was almost entirely lost. The discrepancies in the reckoning of the calendar, the defects in the mensuration of forms, and in the forms of the square, circle, straight, and curved lines, and the disorder occasionally appearing in such common standards as the *dou*, *sheng*, *shao*, *he*[6], *hao*, *li*, *si*, and *hu*[7], prove to us the discontinuation of the development of the ancient science.

Zhu Songting [8] of Yanshan is well-known as a great mathematician

having traveled throughout the country for more than 20 years. The number of people coming from all directions was increasing daily; therefore the master wrote this book in order to reveal the secret of the *Nine Chapters* to his pupils. The book is divided into three parts in which there are 24 sections with 288 problems. It is given the title of *Si Yuan Yu Jian* (Jade Mirror of the Four Unknowns).

His method of solving equations is by putting the *yuan qi* (元气, element of void)in the centre, the element *tian* (天, heaven or sky) at the bottom, the element *di* (地, earth) on the left, the element *ren* (人, man) on the right, and the element *wu* (物 , thing) at the top, by moving the positive and negative terms from the top to the bottom, from the right to the left, by interchanging and alternating their positions, and by many other different ways of arrangement of the terms. The book is used for the solution of problems of excess and deficiency, of implicity of numbers, of equations containing positive and negative terms, and of evolution (which includes solution of higher equations). It is profoundly wonderful [9] . It extends the principles which were founded by the ancient scholars. By the concentration of many elements into one, by controlling the *san cai* (三才), or three talents, under the great extremity, and by multiplication and division, addition and subtraction it reaches out to the great depth and far distance. It is a book of science in itself.

Now the science of mathematics is considered very important and an examination of the subject will appear gradually. The book of the master will, therefore, be of great benefit to the people of the world. The knowledge for the investigation of things, the development of intellectual power, the way of controlling the kingdom and of ruling even the whole world, can be obtained

by those who are able to make a good use of the book. Ought not those who have great desire to be learned take this with them and study it with great care?

In the day of *shang yuan* of the year *gui mao*[10] during the reign of *Da De*, written by Qian Jin Shi Mo Ruo of Linchuan.

【Notes】

[1] There is an old tradition in China that ancient sage, Fu Xi (2852 B. C.) saw a dragon horse emerge from the Yellow River with the *tu* (chart) on its back. Tradition also states that Yu, the first emperor of the Xia dynasty observed the *shu* (book) on the back of the sacred tortoise. (C)

[2] Huang Di, an legendary emperor in the 26th century B. C., caused his minister Li Shou to write the *The Nine Chapters of Mathematical Procedures*. (C)

[3] *Da yan* (great extension) is a rule for the solution of indeterminate problems. The general principle is given by Qin Jiushao in the *Shu Shu Jiu Zhang*. (C)

[4] Lixiang, calculation of astronomical terms such as the reckoning of the calendar. (C)

[5] Wan, yi, and zi, that is 10,000, 100,000,000, and 1,000,000,000,000, are numbers mentioned in the collection of classic poems, *Shi Jing*. (C)

[6] Heaped Measure, see Appendix: Tables of Measures Used by Zhu Shijie in His *Introduction to Mathematical Studies*. (C)

[7] Linear Measure, see Appendix: Tables of Measures Used by Zhu Shijie in his *Introduction to Mathematical Studies*. (C)

[8] Zhu Songting is another name of Zhu Shijie. (C)

[9] The original text changed the *xuan* (玄) into *yuan* (元) in order to avoid

Emperor Kang Xi's name. In the following text, we directly return the character *yuan* to the former character *xuan*, and don't annotate anymore. （G）

[10] The year gui mao means 1303. （G）

松庭先生《四元玉鉴》后序

[元] 祖 颐

【原文】

《黄帝九章》以降，算经多矣，不可枚举。唐宋设明算科，立法取士，不出《九章》《周髀》《海岛》《孙子》《张丘建》[1]《夏侯阳》《五曹》《五经算》《缉古》《缀术》数家而已。然天、地、人、物四元[2]罔有云及一者。厥后平阳蒋周撰《益古》，博陆李文一撰《照胆》，鹿泉石信道撰《钤经》，平水刘汝谐撰《如积释锁》，绛人元裕细草之[3]，后人始知有天元也。平阳李德载因撰《两仪群英集臻》，兼有地元，霍山邢先生颂不高弟刘大鉴润夫撰《乾坤括囊》，末仅有人元二问。

吾友燕山朱汉卿先生演数有年，探三才之赜，索《九章》之隐，按天、地、人、物立成四元，以元气居中，立天——勾，地——股，人——弦，物——黄方，[4] 考图明之。上升下降，左右进退，互通变化，乘除往来，[5] 用假象真，以虚问实，错综正负，分成四式。必以寄之、剔之，余筹[6]易位，横冲直撞，精而不杂，自然而然，消而和会，以成开方之式也。书成，名曰《四元玉鉴》。厘为三卷，以象三才；四元，以象其时；分门二十有四[7]，以象其气；立问二百八十有八，假象周天之数。[8] 玉者[9]，比汉卿之德术，动则其声清越以长，静则孚尹旁达而不有阴翳；鉴者[10]，照四元之形象，收则其缊昭彻而明，开则纵横发挥而曲尽妙理矣。

汉卿，名世杰；松庭，其自号也。[11] 周流四方，复游广陵。踵门而学者云集。大德己亥编集《算学启蒙》，赵元镇已与之版而行矣。

元镇者，博雅之士也。惠然又备己财鸠工绣梓，俾之并行于世，前成始而今成终也。好事之德，奚可量哉！二书相为表里，不其韪欤？

属余为引。余详观之，有素所未尝接于耳目者。不用而用以之通，非数而数以之成，由是而知有数皆从无数中来，高迈于前贤，能尽其妙矣。明算君子据余言试为细草，然后知诚而不妄也。于是乎书。

大德登科[12]二月甲子滹纳心斋祖颐季贤父序

【注释】

[1] 丘，原刻本避孔子名讳改作“邱”。（郭）

[2] 天、地、人、物被用来代表未知数，如 x，y，z，以及 w。（陈）

[3] 细草之，罗士琳改作“之细草”，不妥。（郭）

[4] 在直角三角形中，与水平垂直的边为股，水平的边为勾，斜边为弦。两条直角边的和与斜边的差为黄方。（陈）

[5] 项的转换和化简。（陈）

[6] 用来计算的小棍。（陈）算筹分纵横两式：

	1	2	3	4	5	6	7	8	9
纵式	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
横式	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

个、百、万……位用纵式，十、千、十万……用横式，纵横相间，用空位表示0，可以表示任何一个数字，也可以表示分数、小数、方程和方程组。（郭）

[7] 中国人将一年分为24个节气。（陈）

[8] 周天被分为360度，为地球的轨道。（陈）288为周天之数，待考，$288 = 24 \times 12$，其中24为一太阳年的节气数，12为月数，可为一说。（郭）

[9] 玉为翡翠，宝石，或者其他贵重的石头。翡翠具有许多优良的品质，声音清越便是其中之一，所以中国人特别赞美它。翡翠被用来作为垂饰或装饰，且被看作是一种好的征兆。（陈）

[10] 鉴，即镜子。（陈）

[11] 中国人一般有两或三个名字。第一个是由父母或老人用的，后两个是由朋友使用的。(陈)

[12] 登科，戴煦认为系"癸卯"(1303年)之误。(郭)

【今译】

自《黄帝九章算经》以来，算经很多，不可以枚举。唐、宋两朝设立明算科，立法取士，没有超出《九章算术》《周髀算经》《海岛算经》《孙子算经》《张丘建算经》《夏侯阳算经》《五曹算经》《五经算术》《缉古算经》《缀术》数家的范围。然而，天、地、人、物四元，没有任何著作涉及其中一个。自那以后，平阳蒋周撰《益古集》，博陆李文一撰《照胆》，鹿泉石信道撰《钤经》，平水刘汝谐撰《如积释锁》，绛人元裕为之作细草，后人才知道有天元术。平阳李德载通过撰《两仪群英集臻》，兼有天元、地元的二元术，霍山邢颂（字不高）之弟刘大鉴（字润夫）撰《乾坤括囊》，其末尾仅有使用天元、地元、人元的三元术的两个问题。

我的朋友燕山朱汉卿先生研究数学有年头了。他探讨天、地、人三才的深奥，求索《九章算术》的隐秘，按照天、地、人、物立成四元，将元气放置在中央，设立天元作为勾，地元作为股，人元作为弦，物元作为黄方，考察图形以阐明之。各元能够上升下降，左进右退，使之互相通达变化，能进行乘除运算，并通过用假象真，以虚问实，正负交错，分列成四个四元式。再必须经过寄式，剔而消之，并使剩余的筹式左右易位，横冲直撞，精而不杂，自然而然，消元后和谐会通，以化成一个开方式。书完成之后，命名为《四元玉鉴》。分为三卷，以象征天、地、人三才；四元，以象征春、夏、秋、冬四时；分成二十四门，以象征二十四节气；设立二百八十八问，借以象征周天之数。玉者，比拟汉卿的

德术，动则其声音清越而悠长，静则晶莹剔透而没有阴霾；鉴者，照耀四元之形象，收则其渊奥透彻而明亮，开则纵横捭阖尽情发挥而曲尽妙理。

汉卿，名字叫世杰；松庭，是他的号。他周游东西南北，又到广陵。登门造访跟他学习的人云集。大德己亥年（1299年）编著《算学启蒙》，承蒙赵元镇已给他刻版而行世了。

元镇先生，是一位博雅之士，慨然用自己的钱财召集工匠，将这部书精美地刻版印刷，使之与《算学启蒙》并行于世。《算学启蒙》是开始，而《四元玉鉴》是完美的终结。做好事的品德，怎么可以度量呢？这两部书互相为表里，难道不是值得赞美的吗？

汉卿嘱我作一引言。我详细地阅读了此书，发现有素来未见未闻的内容。看来没有用而大有用处，是因为它能够通达万物，看来没有数而归为数学，是因为它可以成就众理，由此可见，有数都是从无数中来的，而汉卿先生更是超越前贤，曲尽数学的奥妙。通晓数学的君子，根据我的话，试着为此书做细草，然后就知道我的话是真诚的而不是虚妄之言。于是乎写了上面的文字。

大德登科二月甲子濼纳心斋祖颐季贤父序

THE LATTER INTRODUCTION OF *JADA MIRROR OF THE FOUR UNKNOWS*

by Zu Yi

After the *Nine Chapters* of Huang Di was published, numerous books on mathematics appeared. The *Nine Chapters*, *Zhou Bi*, *Hai Dao*, *Sun Zi*, *Zhang Qiu Jian* [1], *Xia Hou Yang*, *Wu Cao*, *Wu Jing Suan*, *Ji Gu*, and *Zhui Shu* were the books used in the examinations of mathematics for the selection of scholars to investigate the science of mathematics. But the four elements, *tian*, *di*, *ren* and *wu* [2] are not mentioned in any of these books. Later we find the element *tian* in the book *Yi Gu*, composed by Jiang Zhou of Pingyang, in the book *Zhao Dan* by Li Wenyi of Bolu, in the *Qian Jing* by Shi Xindao of Luquan, in the *Ru Ji Shi Suo* by Liu Ruxie of Pingshui, and in the work of Yuan Yu of Jiang [3]. It is from these books that Li Dezai of Pingyang obtained the materials for his book *Liang Yi Qun Ying Ji Zhen*, which contains also the element *di*. In the *Qian Kun Kuo Nang* by Liu Dajian Run Fu, the brother of Xing Song (his another name is Xing Bugao) of Huoshan, there are two problems at the end of the book with the element *ren*.

My friend Zhu Hanqing of Yanshan has for many years shown a deep interest in mathematics. He traces out the vestiges of the three talents searches into the secret of the *Nine Chapters*, and considers the four elements *tian*, *di*, *ren* and *wu*. He puts the *yuan qi*, void, in the middle and lets *tian* be the *gou*, *di* the *gu*, *ren* the *xian*, and *wu* the *huang fang*.[4] By the aid of geometric figures he explains their relations. By moving the expression from top to bottom, from right to left, by applying multiplication and division, by various

methods in arranging the terms,[5] by assuming the unreal for the real, by using the imaginary for the true, by using the signs positive and negative, by keeping some and eliminating others and then changing the position of the sticks[6], and by attacking from the front or from one side, as shown in the four examples, he finally works out the expression of evolution in a profound yet natural manner. Now that the book is completed he calls it *Si Yuan Yu Jian*. It contains three books corresponding to the three talents, four elements corresponding to the four seasons, 24 sections corresponding to the 24 festivals[7], and 288 problems pretending to correspond to the *zhou tian*.[8] The *yu*[9] may be considered the virtuous knowledge of Hanqing: when in action it gives a very clear and deep tone and when at rest it shines out without restraint in beautiful rays in all directions. The *jian*[10] may be considered of use in reflecting the phenomena of the four elements; when hidden it keeps its brightness and when exposed it shines in all directions.

Hanqing's name is Shijie and his *hao* is Songting.[11] He has traveled through the country and at present is again sojourning in Guangling. People, like clouds, come from the four directions to meet at his gate in order that they may learn from him. In the year *ji hai* in the reign of Da De he composed the book *Suan Xue Qi Meng*, which was published by the encouragement of Zhao Yuanzhen, a well-educated gentleman, who has given a part of his wealth to hire workers for the cutting of the plates for the book in order to make it known to the world. He has thus assisted in the work from the beginning, and will continue his unlimited generosity until the work is completed.

These two books are like two sides of the same thing, the inner and the

outer, each depending on the other.

When I was asked to write an introduction for this book I read it through with great care and found that there were many things that I had never seen or heard of before. By not using "yet" it is used, by not using a number the number required is obtained. Hence I know that existing quantities come from non-existing quantities. This profound work is therefore exceedingly progressive as compared with the work of ancient mathematicians. Those who have an interest in the subject may prove my words by working out the problems in this book, thus finding the truth of my statement.

Written by Zu Yi Ji Xian Fu

In Hu Na Xin Zhai during the day *jia zi* of the second moon of the year *deng ke* [12] in the reign of *Da De*.

【 Notes 】

[1] The original text changed the *qiu* (丘) into *qiu* (邱) in order to avoid Confucius' name. (G)

[2] *Tian, di, ren, wu* (heaven, earth, man, and things) are used as unknown quantities like x, y, z, and w. (C)

[3] It is not appropriate that Luo Shilin changed *xi cao zhi* into *zhi xi cao*.(G)

[4] In a right triangle the vertical side is called the *gu* (leg), the horizontal side the *gou* (hook), and the hypotenuse the *xian*. The difference between the sum of the two legs and the hypotenuse is called the *huang fang*. (C)

[5] By transposing terms and simplifying. (C)

[6] Sticks used for reckoning. (C) The reckoning-sticks were put in two forms.

One was vertical, the other was horizontal.

	1	2	3	4	5	6	7	8	9
the vertical form	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
the horizontal form	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

The vertical form was used for units, the horizontal form for tens, the vertical form for hundreds, the horizontal form for thousands, and so on. The vertical alternated with the horizontal. 0 was expressed by vacancy. They could express a numeral. And they could express a fraction, a decimal fraction, an equation, and a group of equations. (G)

[7] The Chinese divide the year into 24 festivals as eqinoxes, solstices, etc. (C)

[8] The *zhou tian* (the circumference of the great circle of heaven) is divided into 360 degrees. The earth's orbit. (C) The opinion that 288 is the number of the *zhou tian* needs to be studied. $288 = 24 \times 12$. 24 is the number of solar terms in one solar year, and 12 is the number of months. This opinion would be an explanation. (G)

[9] *Yu* means jade, gem, or other precious stones. Jade is particularly prized by the Chinese because of its many desirable qualities, one of which is its resonance. Jade is worn as a pendant or ornament and is considered a good omen. (C)

[10] *Jian* means mirror. (C)

[11] A Chinese often has two or three names, the first is used by parents and older persons, the last two by friends. (C)

[12] Dai Xu pointed out that *gui mao* (1303) was mistaken for *deng ke*. (G)

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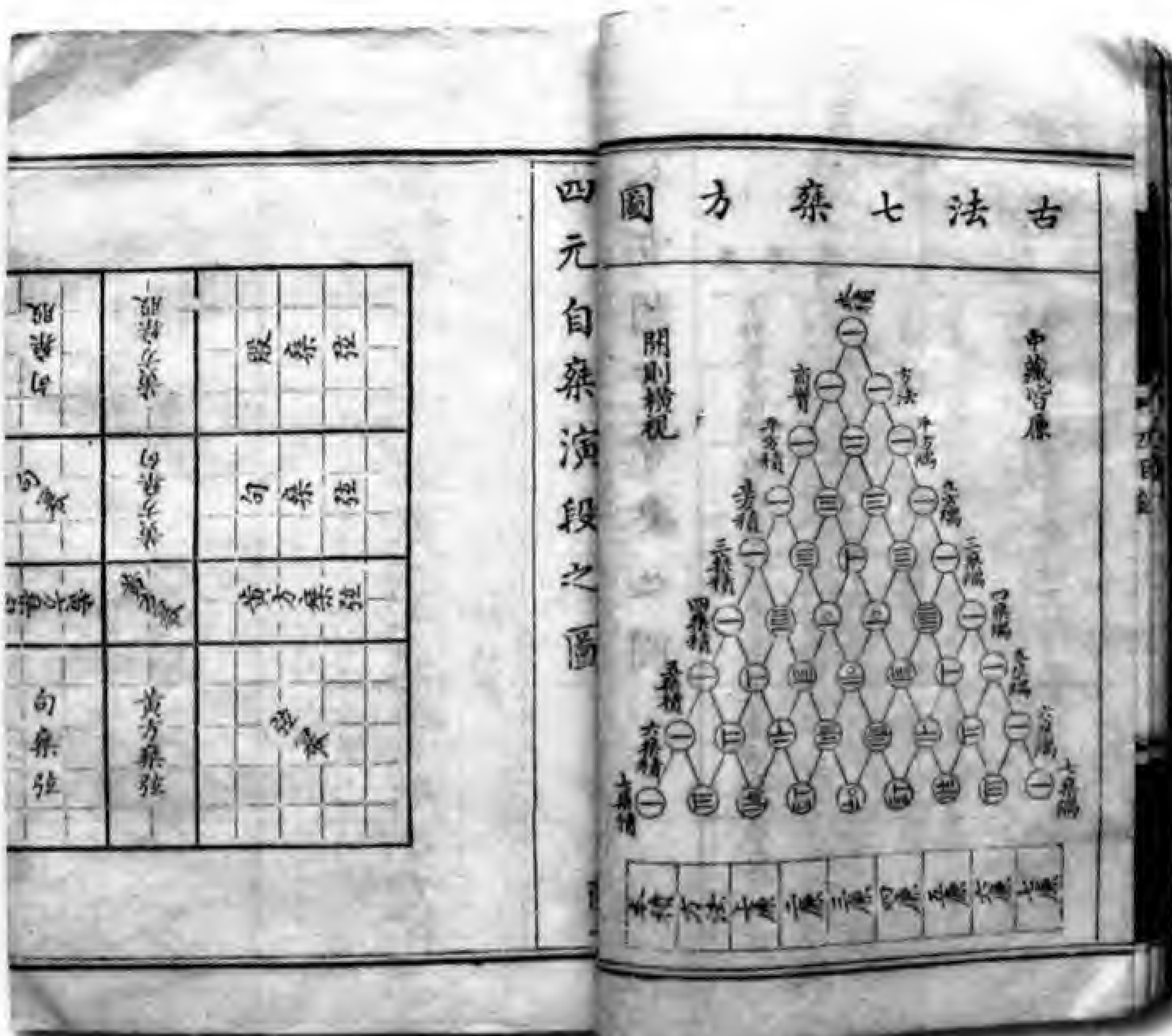


《四元玉鉴》书影〔清道光丙申（一八三六）年刻本〕

今古開方會要之圖

梯法七乘方圖

正者為從負者為益

不動數	商實數	第一等	定實從
方位法	進退一	第二等	除實法
第一廉	進退二	第三等	平方隅
第二廉	進退三	第四等	立方隅
第三廉	進退四	第五等	三乘隅
第四廉	進退五	第六等	四乘隅
第五廉	進退六	第七等	五乘隅
第六廉	進退七	第八等	六乘隅
第七廉	進退八	第九等	七乘隅

CHARTS

Principles of the Ancient and Modern Methods of Evolution (Solution of Equations)

Chart of the Ladder Form of Raising Binomials to the Eighth Power

Positive is called *Zong*, Negative is called Yi			
Ding Shi Wei (定实位) (Position for the Dividend or Absolute)	First Position (第一等)	*Zhi Zhi Shu* (直置数) (Final Result)	*Bu Dong Shu* (不动数) (Number Not Movable)
Chu Shi Fa (除实法) (Divisor)	Second Position (第二等)	*Jin Tui Yi* (进退一) (Forward or Backward one [digit])	*Fang Wei Fa* (方位法) (Position for the Divisor)
Ping Fang Yu (平方隅) (Corner of the Square)	Third Position (第三等)	*Jin Tui Er* (进退二) (Forward or Backward two [digit])	*Di Yi Lian* (第一廉) (1st Complement)
Li Fang Yu (立方隅) (Corner of the Cube)	Fourth Position (第四等)	*Jin Tui San* (进退三) (Forward or Backward three [digit])	*Di Er Lian* (第二廉) (2nd Complement)
San Cheng Yu (三乘隅) (Corner of the 4th Power)	Fifth Position (第五等)	*Jin Tui Si* (进退四) (Forward or Backward four [digit])	*Di San Lian* (第三廉) (3rd Complement)
Si Cheng Yu (四乘隅) (Corner of the 5th Power)	Sixth Position (第六等)	*Jin Tui Wu* (进退五) (Forward or Backward five [digit])	*Di Si Lian* (第四廉) (4th Complement)
Wu Cheng Yu (五乘隅) (Corner of the 6th Power)	Seventh Position (第七等)	*Jin Tui Liu* (进退六) (Forward or Backward six [digit])	*Di Wu Lian* (第五廉) (5th Complement)
Liu Cheng Yu (六乘隅) (Corner of the 7th Power)	Eighth Position (第八等)	*Jin Tui Qi* (进退七) (Forward or Backward seven [digit])	*Di Liu Lian* (第六廉) (6th Complement)
Qi Cheng Yu (七乘隅) (Corner of the 8th Power)	Ninth Position (第九等)	*Jin Tui Ba* (进退八) (Forward or Backward eight [digit])	*Di Qi Lian* (第七廉) (7th Complement)

【陈在新注释】

为了理解这张图表，我们应该知道使用于解方程或者更恰当地说为求不同次幂的方程的根的简略过程。在中国古代，尽管人们在解高次数字方程方面已经能够达到较高的效率，如下文所示，方程这个词却在解题过程中从未被使用过。

古人使用算板与算筹来求解方程。竹制的算筹分红、黑两种。红色的算筹代表正数，黑色的代表负数。只有到了他们开始使用纸和墨来解题的时候，负号才被创造出来。

在含有一个未知数的方程中，天总是被用来表示未知数。如同现在所使用的 x。虽然中国人已经将天设定为未知数，但它从未在方程中出现。它的系数与它的幂的系数正如我们现在于综合方法中所使用的那样使用。例如，方程 $x^3+4x^2+3x-64=0$，以中算符号表示为

即

$$\begin{array}{r} -64 \\ 3 \\ 4 \\ 1 \end{array}$$

含有一个未知量的方程的一般形式为：

$kx^n+\cdots+dx^3+cx^2+bx+a=0$

以中算符号表示为：

$$\begin{array}{c} a \\ b \\ c \\ d \\ \vdots \\ k \end{array}$$

In order to understand the diagram we ought to know simpler processes used in solving equations or rather the evolution of expressions of different degrees. The word equation was never used by the Chinese in ancient times though they attained a high degree of efficiency in solving numerical equations of higher degree as the following pages show.

They worked by means of a calculating board and *suan chou* (calculating rods). The bamboo calculating rods were red and black, the red indicating positive numbers and the black negative. It was only when they began using paper and ink for the solution of problems that the negative sign was created.

In equations in one unknown quantity the element *tian* (天) was always used for the unknown, as x is used in modern times. Although the Chinese had the assumed unknown *tian* (天), it never appeared in the equation. Its coefficients and the coefficients of its power were used as we now use them in the synthetic method. For example, the equation $x^3 + 4x^2 + 3x - 64 = 0$, written in the Chinese notation, was

⊥ ||||̸		-64
|||	meaning	3
||||		4
			1

The general form of an equation in one unknown letter

$kx^n + \cdots + dx^3 + cx^2 + bx + a = 0$

written in the Chinese notation was

$$\begin{array}{c} a \\ b \\ c \\ d \\ \vdots \\ k \end{array}$$

由于它是一个表达式而不是一个方程，人们总将它理解为等于0，并称之为开方式。

在一次表达式中，未知数的系数被称作除实法，常数项为实。在高于一次的表达式中，未知数一次项的系数被称作方。

在二次表达式中［现代形式为 $ax^2+bx+c=0$］。a 被称作隅或者平方隅，b 为方，c 为实。

在高于二次的高次表达式中，第一项被称作隅，或者以表达式次数限制的隅。如果是一个三次表达式，就是立方隅，如果是一个四次表达式，就是四次隅。最后一项总被称作实，倒数第二项为方。在第一项与倒数第二项之间的项被称作“廉”，并以它们的次序加以区分，即：第一廉，第二廉，等等。第一廉也被称为上廉，末廉为下廉。

决定根的位数的过程被称作步法。在运算过程中，尽管古人知道负根能够被得到，但是为了得到其认为是有理根的正根，总是采用两个符号相反的连续项。在求根过程之中，当方向前（左）移动一位，商（根）将包含两位，等等。

每移动一次被称作一步。因此，商的位数总是比步数多一。常数项的位置保持固定。当其他项中的任何一项移动时，所有项都将移动。例如，如果方移动一步，廉与隅都移动一步，反之亦然；如果方移动两步，表达式的其他项也移动两步，反之亦然。在每一步中，当第一廉移动两位，第二廉移动三位，第三廉移动四位时，方移动一位，等等。

为了更清楚地了解这些，我们不妨以如下的例子加以阐释。

例1. 解 $2x=4$ 或者 $4-2x=0$

以中算符号表示为

$$\begin{array}{c} 4 \\ -2 \end{array}$$

4是实，-2 为除实法。

2是表达式的商或者根。

$$2\times(-2)=-4$$

This expression, always understood equal to zero, is called *kai fang zhi shi* (expression of evolution) since it is an expression and not an equation.

In an expression of the first degree the coefficient of the unknown is called the *chu shi fa* (除实法) and the absolute term the *shi* (实, dividend). In an expression of higher degree than the first the coefficient of the first power of the unknown is called the *fang* (方, side).

In an expression of the second degree (modern form $ax^2 + bx + c = 0$), a is called yu(隅, corner)or *ping fang yu*(平方隅, corner of a square), b *fang*, and c *shi*.

In an expression of higher degree than the second the first term is called *yu*, or *yu* modified by the degree of the expression, that is, cubic *yu* if it is a cubic expression, or the fourth power of *yu* if it is an expression of the fourth degree. The last is always called *shi* and the second from the last *fang*; the terms between the second from the last and the first are called *lian* (廉, complementary), and are distinguished by their order; that is, first *lian*, second lian, and so on. The first *lian* is also called the upper *lian* and the last *lian* the lower *lian*.

The determination of number of digits in a root is called *bu fa* (步法, method to step or method of procession). In procession they always took two consecutive terms of opposite signs in order to obtain the positive root which they considered the rational root although they knew negative roots could be obtained. In procession when the *fang* moves forward (to the left) one digit the quotient (or root) will contain two digits and so on.

Each move is called a step. Thus the number of digits in the *shang*(商, quotient or root) always exceed the number of steps by one. The position of the absolute term remains fixed. When one of other terms move all move. For instance, if the *fang* (方) moves one step the *lian* and the *yu* all move one step and vice versa; if the *fang* moves two steps the other terms of the expression move two steps and vice versa. In each step, the *fang* moves one digit while the first *lian* moves two, the second three, the third four, and so on.

These steps may be made more clear by the following examples.

Example 1. Solve $2x = 4$ or $4 - 2x = 0$

We have in the Chinese form

$$\begin{array}{r} 4 \\ -2 \end{array}$$

4 is the *shi* and -2 is the *chu shi fa*.

2 is the quotient or root of expression.

$$2 \times (-2) = -4$$

加 $4+(-4)=0$

值得注意，这个表达式，并不需要任何解题步骤。

例2. 解 $2x=40$

或者

$$\begin{array}{c} 40 \\ -2 \end{array}$$

在解题过程中，方（-2）必须移动一步（每步一位）。我们得到

$$\begin{array}{rl} 40 & \text{实} \\ -2 & \text{除实法} \end{array}$$

这表示商有两位。它的第一位是2。

$$2\times(-2)=-4$$

加 $4+(-4)=0$ 此即第二位的值。

因此，20是方程的根。

例3. 解 $x^2=64$

或者

$$\begin{array}{r} 64 \\ 0 \\ -1 \end{array}$$

这是一个没有方的二次表达式。

在此题中，我们使用隅来步实。在一个二次表达式的每一步中，隅要移动两位。在此例中，隅移动的位数将超出实的位数，因而我们不需要步之，根仅有一位。

用中算形式，我们得到

$$\begin{array}{rl} \underline{8} & \text{根} \\ 64 & \text{实} \\ \underline{-64} & \\ 0 & \\ 0 & \text{方} \\ \underline{-8} & \\ -8 & \\ -1 & \text{隅} \end{array}$$

Adding $4 + (-4) = 0$

Note that in this expression we do not need procession.

Example 2. Solve $2x = 40$

or

$$\begin{array}{c} 40 \\ -2 \end{array}$$

In procession the *fang*(-2)has to move one step(one digit each step). We have

$$\begin{array}{ll} 40 & \textit{shi} \\ -2 & \textit{chu shi fa} \end{array}$$

Which indicates that the quotient has two digits, the first of which is 2.

$$2 \times (-2) = -4$$

Adding $4 + (-4) = 0$ the value of the second digit.

Hence 20 is the root of the equation.

Example 3. Solve $x^2 = 64$

or

$$\begin{array}{c} 64 \\ 0 \\ -1 \end{array}$$

a quadratic expression with the *fang* absent.

In this problem we use the *yu* to step the *shi*. In each step of a quadratic expression the *yu* moves two digits. In this case the *yu* would exceed the shi, therefore we do not need procession and the root has only one digit.

In the Chinese form we have

$$\begin{array}{rl} \underline{8} & \text{root} \\ 64 & \textit{shi} \\ \underline{-64} & \\ 0 & \\ 0 & \textit{fang} \\ \underline{-8} & \\ -8 & \\ -1 & \textit{yu} \end{array}$$

用现代形式，我们得到

$$\begin{array}{rrrl} -1 & 0 & 64 & (8，根 \\ & \underline{-8} & \underline{-64} & \\ & -8 & 0 & \end{array}$$

例4. 解 $-x^3 + 392x^2 + 3185x + 6000 = 0$

或者

$$\begin{array}{r} 6000 \\ 3185 \\ 392 \\ -1 \end{array}$$

在这个表达式中，6000被称作实，3185为方，392为廉，−1为隅。在解题过程中，为了得到一个正根，我们必须以隅−1，来步廉392。为了使它与廉的第一位对齐，必须将它移动两步。

我们得到

$$\begin{array}{rl} 6000 & 实 \\ 3185 & 方 \\ 392 & 廉 \\ -1 & 隅 \end{array}$$

这里仅是解题过程的法则。解题方法与前述问题的解法相似。

例5. 解 $-4x + 104 = 0$

或者

$$\begin{array}{r} 104 \\ -4 \end{array}$$

程序。

$$\begin{array}{rl} 2 & 根的第一位数 \\ 104 & 实 \\ \underline{-8} & \\ 24 & 下一个实 \end{array}$$

In modern form we have

$$\begin{array}{ccc} -1 & 0 & 64 \text{ (8 , root} \\ & \underline{-8} & \underline{-64} \\ & -8 & 0 \end{array}$$

Example 4. Solve $-x^3 + 392x^2 + 3185x + 6000 = 0$

or

$$\begin{array}{r} 6000 \\ 3185 \\ 392 \\ -1 \end{array}$$

In this expression 6000 is called the *shi*, 3185 the *fang*, 392 the *lian*, and −1 the *yu*. In procession we have to take the *yu*, −1, to step the *lian*, 392, in order to get a positive root. It must make two steps in order to be even with the first digit of the *lian*.

We have

6000	*shi*
3185	*fang*
392	*lian*
−1	*yu*

Only the law of procession is shown. The solution is similar to that of the preceding problem.

Example 5. Solve $-4x + 104 = 0$

or

$$\begin{array}{r} 104 \\ -4 \end{array}$$

Process.

2	the first digit of the root
104	*shi*
$\underline{-8}$	
24	next *shi*

-4	除实法
6	根的最后一位数
24	实
$\underline{-24}$	
0	
-4	除实法

（注：–4 向后移动一步）

因此，所求的根是26。

例6. 解 $x^2 + 44x - 3116 = 0$

或者

$$\begin{array}{r} -3116 \\ 44 \\ 1 \end{array}$$

通过以方44步实 –3116，我们发现根有两位。在移动过程中，当隅（1）移动两位时，方（44）仅移动一位，因而得到

-3116	实
44	方
1	隅

程序：

	3	根
	-3116	实
加74的3倍，	$\underline{222}$	
	-896	下一个实
加1的3倍，	44	方

−4 *chu shi fa*

6 the last digit of the root

24 *shi*

$\underline{-24}$

0

−4 *chu shi fa.*

(Note that −4 steps backward one step) Hence the required root is 26.

Example 6. Solve $x^2 + 44x - 3116 = 0$

or

$$\begin{array}{r} -3116 \\ 44 \\ 1 \end{array}$$

By using the *fang*, 44, to step the *shi*, −3116, we find the root has two digits. In stepping the *fang* (44) moves only one digit while the *yu* (1) moves two digits, thus giving

−3116	*shi*
44	*fang*
1	*yu*

Process.

	3	root
	−3116	*shi*
adding 3 times 74,	$\underline{222}$	
	−896	next *shi*
adding 3 times 1,	44	*fang*

　　　　3

　　74

加 1 的 3 倍，　3

　　104　下一个方

　　1　隅

　　8　根的最后一位

　　−896　实

加　896

　　0

　　104　方（向后移一步）

加 1 的 8 倍，　8

　　112

　　1　隅（向后移一步或两位）

通过检验可以看到，根的第一位数是3。以方44开始，加隅的3倍，为3，我们得到74。

取下一项，即实，　−3116

然后加 74 的 3 倍，　222

我们得到下一个实，　−896

再取 74，然后加隅的 3 倍，即 3，我们得到

　　74

　　3

　　104 为下一个方

已经得到下一个方与下一个实，我们接下来计算根的第二位数字。新形式为

　　−896　实

　　104　方

　　1　隅

	3	
	74	
adding 3 times 1,	3	
	104,	next *fang*
	1	*yu*
	8	last digit of root
	–896	*shi*
Adding	896	
	0	
	104	*fang* (one step backward)
adding 8 times 1,	8	
	112	
	1	*yu*,(one step or two digits backward)

By inspection we see that the first figure of the root is 3. Beginning with the *fang*, 44, and adding 3 times the *yu*, 3, we have 74.

Taking the next term, the *shi*,	–3116
and adding 3 times 74,	222
we have the next *shi*,	–896

Again taking 74 and adding 3 times the *yu*, 3, we have

74

3

104 for the next *fang*.

Having obtained the next *fang* and next *shi* we proceed to obtain the second figure of the root. The new form is

–896	*shi*
104	*fang*
1	*yu*

注：在新的形式中，方与隅已经向后移动一步，即向右移动一步。通过检验，我们知道根的下一位数字大约是8。

以新的方开始，	104
加上隅的8倍，	8
我们得到，	112
选取实，	−896
然后，加112的8倍，	896
我们得到，	0

因此，38是所求的根。

例7. 解 $x^3 - 37x^2 + 37x - 36 = 0$

或者

$$\begin{array}{r} -36 \\ 37 \\ -37 \\ 1 \end{array}$$

以隅1步廉−37，我们发现根有两位。单独考虑该廉与隅，隅必须移动一位，因而整个表达式为

$$\begin{array}{r} -36 \\ 37 \\ -37 \\ 1 \end{array}$$

廉与方均移一步，实不动。正如前文的方法所示，这使得方移一位，廉移两位，隅移三位。

3	根
−36	实
−519	
−5226	下一个实

Notice that in the new form the *fang* and the *yu* have moved one step backward, that is, one step to the right.

By inspection we know that the next figure of the root is about 8.

Beginning with the new *fang*,	104
Adding 8 times the *yu*,	8
We have,	112
Taking the *shi*,	–896
and adding 8 times 112,	896
We have,	0

Hence 38 is the required root.

Example 7. Solve $x^3 - 37x^2 + 37x - 36 = 0$

or

$$\begin{array}{r} -36 \\ 37 \\ -37 \\ 1 \end{array}$$

By using the *yu*, 1, to step the *lian*, –37, we find the root has two digits. In considering the *lian* and the *yu* alone, the *yu* has to move one digit, hence in the whole expression

$$\begin{array}{r} -36 \\ 37 \\ -37 \\ 1 \end{array}$$

the complementary digits, the *lian* and the *fang*, each move one step, the *shi* never moving. This makes the *fang* move one digit, the *lian* two digits, and the *yu* three digits, as shown in the solution.

3	root
–36	*shi*
–519	
–5226	next *shi*

37 方

$\underline{-21}$

−173

$\underline{69}$

517 下一个方

−37 廉

$\underline{3}$

−7

$\underline{3}$

23

$\underline{3}$

53 下一个廉

1 隅

为了得到根的第二位数，我们有

6 根

−5226 实

$\underline{5226}$

0

517 方

$\underline{354}$

871

53 廉

$\underline{6}$

59

1 隅

因此，36是所求的根。

$$\begin{array}{rl}
37 & \textit{fang} \\
\underline{-21} & \\
-173 & \\
\underline{69} & \\
517 & \text{next } \textit{fang} \\
-37 & \textit{lian} \\
\underline{3} & \\
-7 & \\
\underline{3} & \\
23 & \\
\underline{3} & \\
53 & \text{next } \textit{lian} \\
1 & \textit{yu}
\end{array}$$

For obtaining the second digit of the root we have

$$\begin{array}{rl}
6 & \text{root} \\
-5226 & \textit{shi} \\
\underline{5226} & \\
0 & \\
517 & \textit{fang} \\
\underline{354} & \\
871 & \\
53 & \textit{lian} \\
\underline{6} & \\
59 & \\
1 & \textit{yu}
\end{array}$$

Hence 36 is the required root.

该方法的现代形式与霍纳法相同：

$$
\begin{array}{rrrrrrrl}
x^3 & - & 37x^2 & + & 37x & - & 36 = 0 & \\
1000 & - & 3700 & + & 370 & - & 36 & (\,3 \\
 & & \underline{3000} & - & \underline{2100} & - & \underline{5190} & \\
 & & -700 & - & 1730 & - & 5226 & \\
 & & \underline{3000} & & \underline{6900} & & & \\
 & & 2300 & & 5170 & & & \\
 & & \underline{3000} & & & & & \\
 & & 5300 & & & & & \\
1 & + & 53 & + & 517 & - & 5226 & (\,6 \\
 & & \underline{6} & & \underline{354} & & \underline{5226} & \\
 & & 59 & & 871 & & 0 &
\end{array}
$$

故，36是方程的一个根。

例8. 解 $x^2 - 10x + 18.24 = 0$

此方程可变形为 $100x^2 - 1000x + 1824 = 0$

或者

$$
\begin{array}{r}
1824 \\
-1000 \\
100
\end{array}
$$

$$
\begin{array}{rl}
2, & \text{根的第一位数} \\
1824 & \text{实} \\
\underline{-1600} & \\
224 & \text{下一个实} \\
-1000 & \text{方} \\
\underline{200} & \\
-800 & \\
\underline{200} & \\
-600 & \text{下一个方} \\
100 & \text{隅}
\end{array}
\qquad\qquad
\begin{array}{rl}
4 & \text{根的第二位数} \\
224 & \text{实} \\
\underline{-224} & \\
0 & \\
-60 & \text{方} \\
\underline{4} & \\
-56 & \\
1 & \text{隅}
\end{array}
$$

因此，2.4是所求的根。

The solution in modern form is the same as Horner's method

$$x^3 - 37x^2 + 37x - 36 = 0$$

$$\begin{array}{ccccccc}
1000 & - & 3700 + & 370 - & 36 & (& 3 \\
 & & \underline{3000} - & \underline{2100} - & \underline{5190} & & \\
 & & -700 - & 1730 - & 5226 & & \\
 & & \underline{3000} & \underline{6900} & & & \\
 & & 2300 & 5170 & & & \\
 & & \underline{3000} & & & & \\
 & & 5300 & & & & \\
1 & + & 53 + & 517 - & 5226 & (& 6 \\
 & & \underline{6} & \underline{354} & \underline{5226} & & \\
 & & 59 & 871 & 0 & &
\end{array}$$

Hence 36 is one of the roots.

Example 8. Solve $x^2 - 10x + 18.24 = 0$

The equation may be written $100x^2 - 1000x + 1824 = 0$

or

$$\begin{array}{r}
1824 \\
-1000 \\
100
\end{array}$$

2	1st digit of the root	4	2nd digit of the root
1824	*shi*	224	*shi*
$\underline{-1600}$		$\underline{-224}$	
224	next *shi*	0	
−1000	*fang*	−60	*fang*
$\underline{200}$		$\underline{4}$	
−800		−56	
$\underline{200}$		1	*yu*
−600	next *fang*		
100	*yu*		

Hence 2.4 is the required root.

例9. 解 $8x-310=0$

或者 $\begin{array}{r}-310\\8\end{array}$

3	根的第一位数	8	根的第二位数
−310	实	−70	实
24		64	
−70	下一个实	−6	
8	除实法（除数）	8	除实法（除数）

因此，$38\frac{6}{8}$ 或者 $38\frac{3}{4}$ 为所求的根。

例10. 解 $x^2-265=0$

或者 $\begin{array}{r}-265\\0\\1\end{array}$

1	根的第一位数	6	根的第二位数
−265	实	−165	实
1		156	
−165	下一个方	−9	余数
0	方	20	方
1		6	
1		26	
1		6	
2	下一个方	32	
1	隅	1	隅

根的整数部分是 16。

Example 9. Solve $8x - 310 = 0$

or

$$\begin{array}{r} -310 \\ 8 \end{array}$$

3	1st digit of the root	8	2nd digit of the root
−310	*shi*	−70	*shi*
<u>24</u>		<u>64</u>	
−70	next *shi*	−6	
8	*chu shi fa* (divisor)	8	*chu shi fa* (divisor)

Hence $38\frac{6}{8}$ or $38\frac{3}{4}$ is the required root.

Example 10. Solve $x^2 - 265 = 0$

or

$$\begin{array}{r} -265 \\ 0 \\ 1 \end{array}$$

1,	1st digit of the root	6,	2nd digit of the root
−265	*shi*	−165	*shi*
<u>1</u>		<u>156</u>	
−165	next *fang*	−9	remainder
0	*fang*	20	*fang*
<u>1</u>		<u>6</u>	
1		26	
<u>1</u>		<u>6</u>	
2	next *fang*	32	
1	*yu*	1	*yu*

The integral part of the root is 16.

为了获得分数部分的近似值，我们以余数 −9 作为实，32 作为方，1 作为隅。

$$
\begin{array}{rl}
1 & \text{根} \\
-9 & \text{实} \\
33 & \\
32 & \text{方} \\
\underline{1} & \\
33 & \\
1 & \text{隅}
\end{array}
$$

根 1 加方 32 得 33 作为分数的分母，9 作为分子的方法。

因此，$16\frac{9}{33}$ 或者 $16\frac{3}{11}$ 为近似根。

得到分数部分的这种方法被称作借商命分。

例 11. 解 $135x^2 + 4608x - 138240 = 0$

或者

$$
\begin{array}{r}
-138240 \\
4608 \\
135
\end{array}
$$

$$
\begin{array}{rlcrl}
1 & \text{根的第一位数} & \quad & 9 & \text{根的第二位数} \\
-138240 & \text{实} & & -78660 & \text{实} \\
\underline{5958} & & & \underline{76707} & \\
-78660 & \text{下一个实} & & -1953 & \text{下一个实} \\
4608 & \text{方} & & 7308 & \text{方} \\
\underline{135} & & & \underline{1215} & \\
5958 & & & 8523 & \\
\underline{135} & & & \underline{1215} & \\
7308 & \text{下一个方} & & 9738 & \text{下一个方} \\
135 & \text{隅} & & 135 & \text{隅}
\end{array}
$$

To obtain the approximate fractional part we take the remainder −9 as the *shi*, 32 as the *fang*, and 1 as the *yu*.

1	root
−9	*shi*
33	
32	*fang*
1	
33	
1	*yu*

The root 1 is used as the means through which we obtain the term 33 as the denominator of fraction, of which 9 is the numerator.

Hence $16\frac{9}{33}$ or $16\frac{3}{11}$ is the approximate root.

The fractional part obtained by this method is called *jie shang ming fen* (借商命分), or borrow root fractional method.

Example 11. Solve $135x^2 + 4608x - 138240 = 0$

or

$$\begin{array}{r} -138240 \\ 4608 \\ 135 \end{array}$$

1	1st digit of the root	9	2nd digit of the root
−138240	*shi*	−78660	*shi*
5958		76707	
−78660	next *shi*	−1953	next *shi*
4608	*fang*	7308	*fang*
135		1215	
5958		8523	
135		1215	
7308	next *fang*	9738	next *fang*
135	*yu*	135	*yu*

2

−195300 实

195300

0

97380 方

270

97650

135 隅

因此，19.2是方程的根。

得到分数部分的这种方法被称作退商命分，或小数根的求法。

例12. 解 $12x^2 + 49x - 1326 = 0$

或者

−1326

49

12

8	根		8	根
−1326	实	12 × (−166),	−1992	实
1160			1992	
−166	下一个实		0	
49	方		241	方
96			8	
145			249	
96			1	隅
241	下一个方			
12	隅			

因此 $8\frac{8}{12}$ 或者 $8\frac{2}{3}$ 是所求的根。

得到分数部分的这种方法被称作连枝同体，或者之分法。

2	
−195300	*shi*
195300	
0	
97380	*fang*
270	
97650	
135	*yu*

Hence 19.2 is the root.

The fractional part obtained by this method is called the *tui shang ming fen* (退商命分), or decimal root method.

Example 12. Solve $12x^2 + 49x - 1326 = 0$

or

$$\begin{array}{r} -1326 \\ 49 \\ 12 \end{array}$$

8	root		8	root
−1326	*shi*	12 × (−166),	−1992	*shi*
1160			1992	
−166	next *shi*		0	
49	*fang*		241	*fang*
96			8	
145			249	
96			1	*yu*
241	next *fang*			
12	*yu*			

Hence $8\frac{8}{12}$ or $8\frac{2}{3}$ is the required root.

The fractional part obtained by this method is called the *lian zhi tong ti* (连枝同体) (the branch connected with the trunk), or fractional method of a quadratic expression.

例13. 解 $24x^3 - 31x^2 - 55x - 12818 = 0$

或者

$$\begin{array}{r} -12818 \\ -55 \\ -31 \\ 24 \end{array}$$

8	根		1	根
−12818	实	$24^2 \times (-2954)$,	−1701504	实
9864			102918	
−2954	下一个实		−672324	下一个实
−55	方	24×4057,	97368	方
1288			555	
1233			102918	
2824			565	
4057	下一个方		108568	下一个方
−31	廉		545	廉
192			1	
161			555	
192			1	
353			565	
192			1	
545	下一个廉		575	下一个廉
24	隅		1	隅

Example 13. Solve $24x^3 - 31x^2 - 55x - 12818 = 0$

or

$$\begin{array}{r} -12818 \\ -55 \\ -31 \\ 24 \end{array}$$

8	root		1	root
−12818	*shi*	$24^2 \times (-2954)$,	−1701504	*shi*
9864			102918	
−2954	next *shi*		−672324	next *shi*
−55	*fang*	24×4057,	97368	*fang*
1288			555	
1233			102918	
2824			565	
4057	next *fang*		108568	next *fang*
−31	*lian*		545	*lian*
192			1	
161			555	
192			1	
353			565	
192			1	
545	next *lian*		575	next *lian*
24	*yu*		1	*yu*

6	根
−672324	实
672324	
0	
108568	方
3486	
112054	
575	廉
6	
581	
1	隅

因此$8\frac{16}{24}$或者$8\frac{2}{3}$为所求的根。

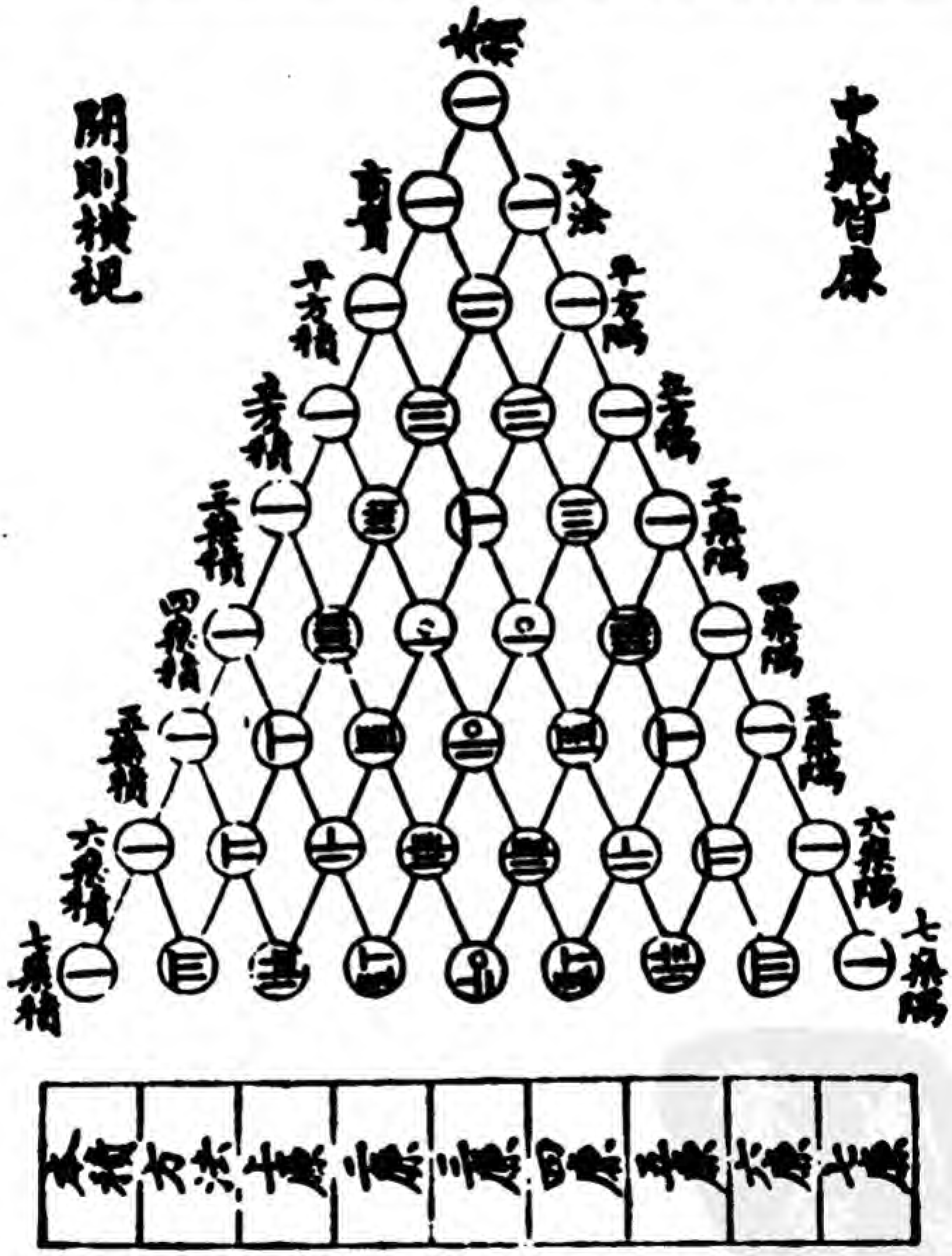
这种方法亦被称之为连枝同体，或者亦称为之分法。

6	root
–672324	*shi*
672324	
0	
108568	*fang*
3486	
112054	
575	*lian*
6	
581	
1	*yu*

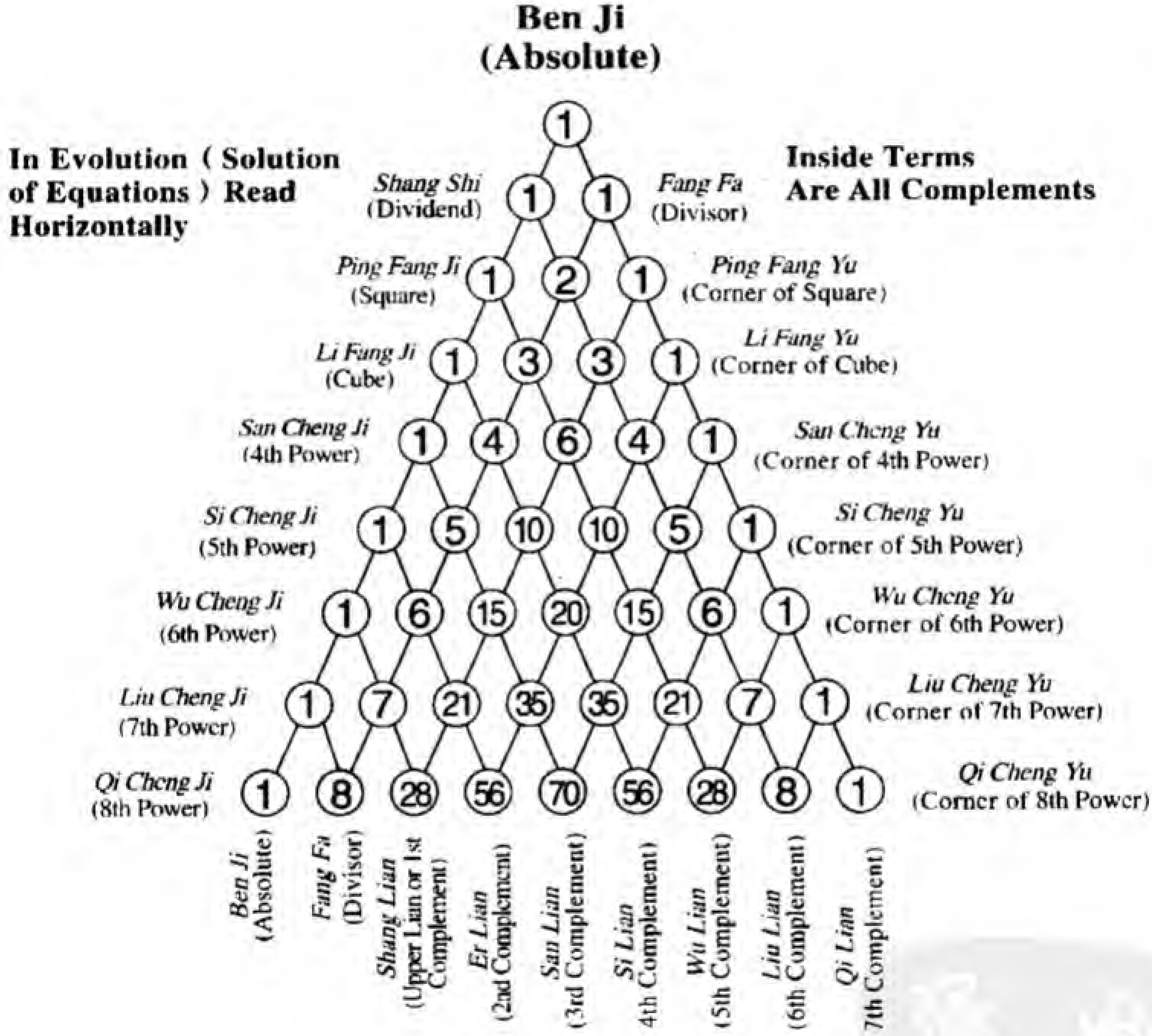
Hence $8\frac{16}{24}$ or $8\frac{2}{3}$ is the required root.

This method is also called the *lian zhi tong ti* (连枝同体)(the branch connected with the trunk), or fractional method of a cubic expression.

古法七乘方圖



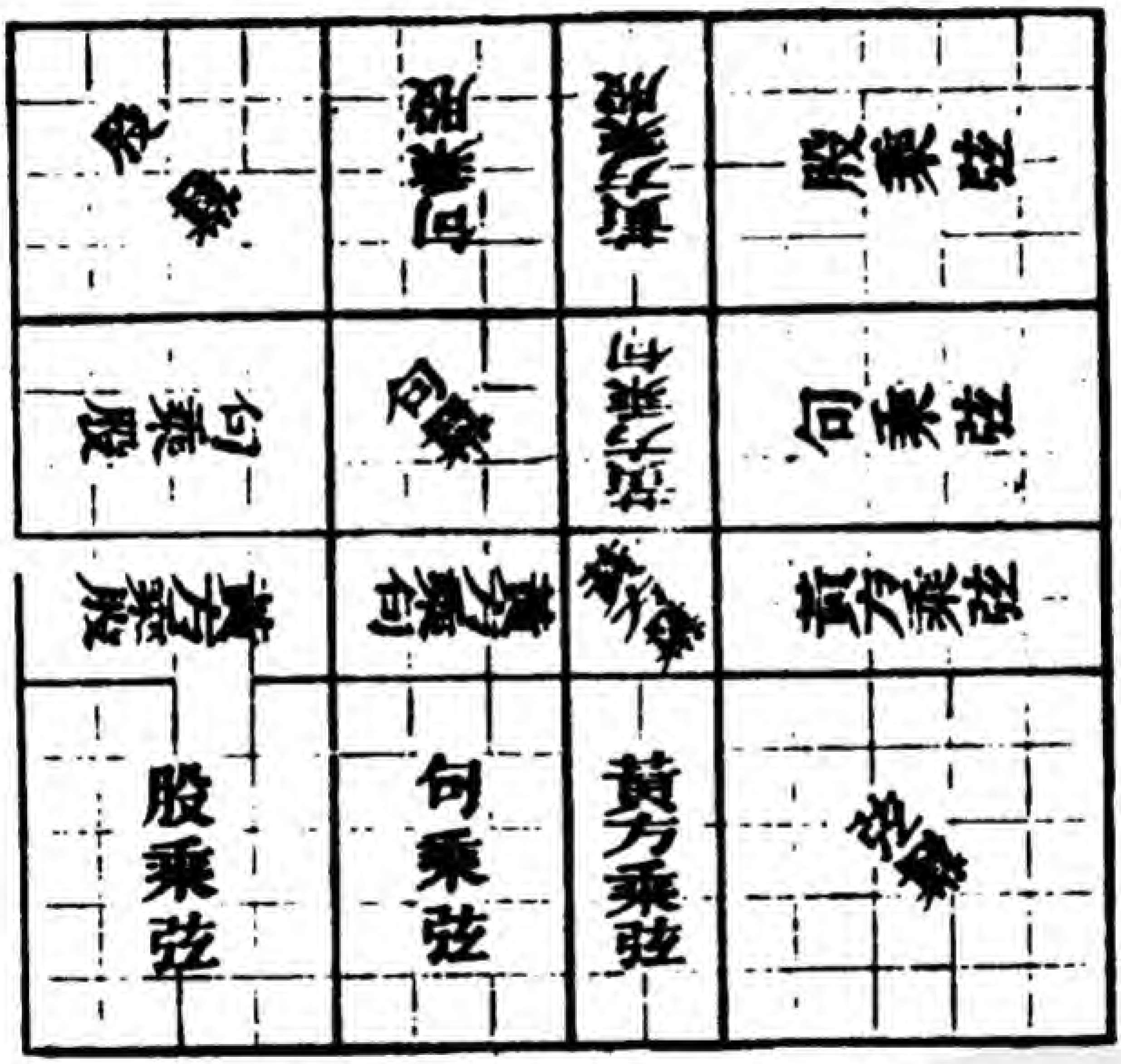
Gu Fa Qi Cheng Fang Tu
(Chart of the Ancient Method of Rais– ing Binomial to the Eighth Power)



【注释】

帕斯卡三角形与古法七乘方图一样。开方的方法也依靠于这个三角形。其中每一项的名称已经在上图中给出。(陈)

四元自乘演段之圖



【Notes】

Pascal's arithmetical triangle is given here as the ancient method of raising binomials to the 8th power. The method of evolution also depends on this triangle. Each term is named as given in the first diagram. (C)

The Square of the Sum of the Four Quantities of a Right Triangle

Square of *Gu*	*Gou* × *Gu*	*Huang Fang* × *Gu*	*Gu* × *Xian*
Gou × *Gu*	Square of *Gu*	*Huang Fang* × *Gou*	*Gou* × *Xian*
Huang Fang × *Gu*	*Huang Fang* × *Gou*	Square of *Huang* Fang	*Huang Fang* × *Xian*
Gu × *Xian*	*Gou* × *Xian*	*Huang Fang* × *Xian*	Square of *Xian*

【原文】

凡习四元者，以明理为务，必达乘除、升降、进退之理，乃尽性穷神之学也。仆立勾三、股四、弦五、黄方二为问，并之，得

　　1
1　太　1
　　1

。自乘为幂，得此式

　　　　1
　2　0　2
　　　2
1　0　太　0　1
　　2
　2　0　2
　　　　1

，共计一十六段，计幂一百九十六步。考图认之，其理显然。

【今译】

凡是学习四元术的人，以明白其原理为首要任务，必须通晓乘、除，四元之升降，筹式的进退的原理，这是尽性穷神的学问。我假设勾 3，股 4，弦 5，黄方 2 以构造问题。并之，得 $x+y+z+u$，自乘为幂，得下式：$x^2+y^2+z^2+u^2+2xy+2xz+2xu+2yz+2yu+2zu$。共计 16 段，幂 196 步。考察图形，道理是显然的。

Those who wish to study the four elements must first have a clear understanding of the theory and also know the rules of multiplication and division, the rules for ascending and descending, and the rules for stepping forward and stepping backward. It is not an easy thing and requires hard thinking. Here I let the base of a right triangle be 3, the leg 4, the hypotenuse 5, and the *huang fang* (yellow square) 2. The sum of these for elements is put in the form

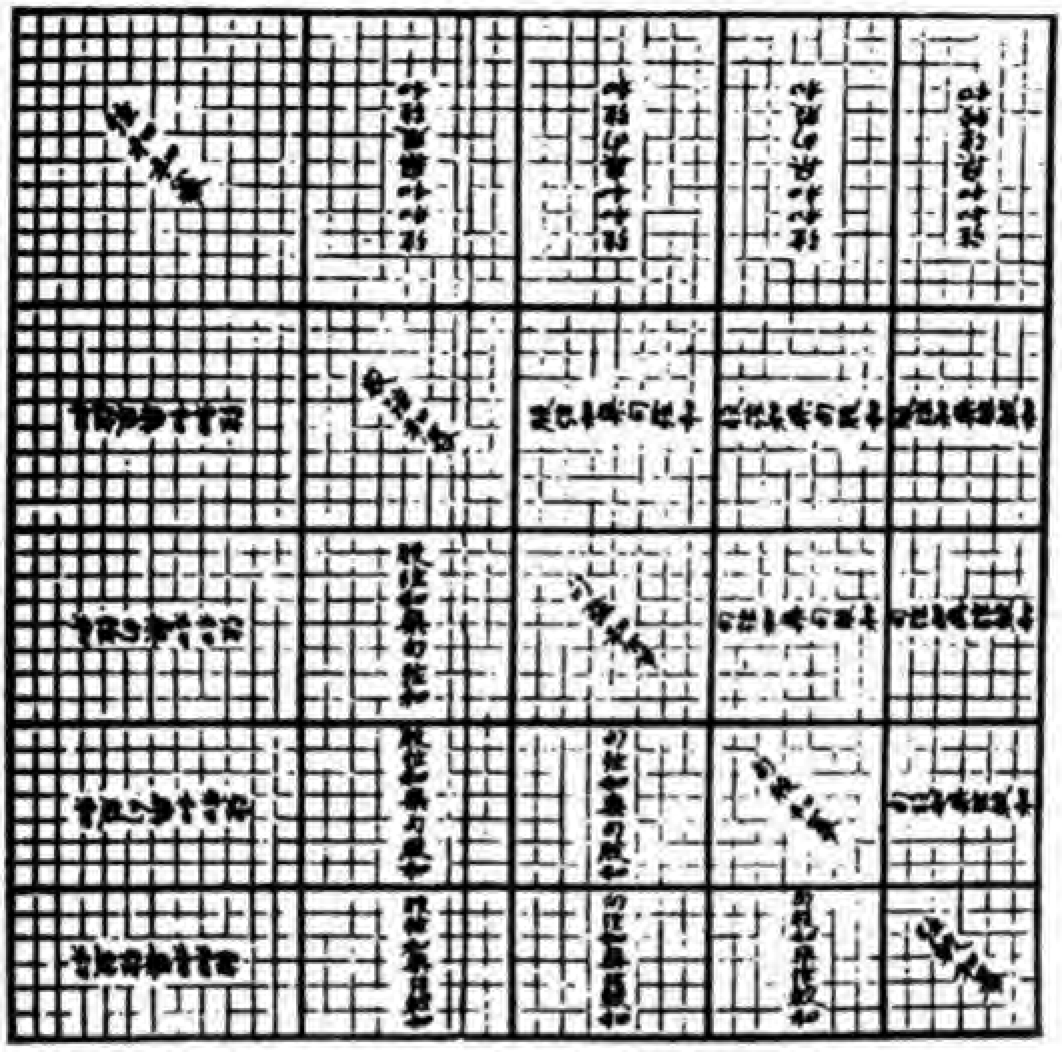
$$\begin{array}{ccc} & 1 & \\ 1 & \text{太} & 1 \\ & 1 & \end{array}$$

whose square is equal to

$$\begin{array}{ccccccc} & & & 1 & & & \\ & 2 & & 0 & & 2 & \\ & & & & 2 & & \\ 1 & 0 & & \text{太} & & 0 & 1 \\ & & 2 & & & & \\ & 2 & & 0 & & 2 & \\ & & & 1 & & & \end{array}$$

Having 16 sections which contain 196 *bu* as shown in this figure.

五和自乘演段之圖



【原文】

凡勾股之术，出于圆方。圆径一而周三，方径一而匝四。伸圆之为勾，展方之为股，共结一角斜弦，适五，勾股之所生也。今言五和者[1]，勾股和、勾弦和、股弦和、弦和和、弦较和。并之，得四十二步。自乘，得一千七百六十四步，共为二十五段也。

【注释】

［1］五和：勾与股之和 $a+b$，勾与弦之和 $a+c$，股与弦之和 $b+c$，弦与勾股和之和 $(a+b)+c$，弦与勾股差之和 $(b-a)+c$。（郭）

【今译】

凡是勾股的方法，都出于圆和方。圆的直径是1，而圆周是3；方的径是1，而方周是4。将圆周展开作为勾，将方周展开作为股，连接起来与勾、股各成一角就成为斜弦，恰好是5，勾股形由此产生出来。现在说的五和就是：勾股和、勾弦和、股弦和、弦和和、弦较和。将它们相加，得到42步。再将其自乘，得到1764步。总共为25段。

The Square of the Five *He*

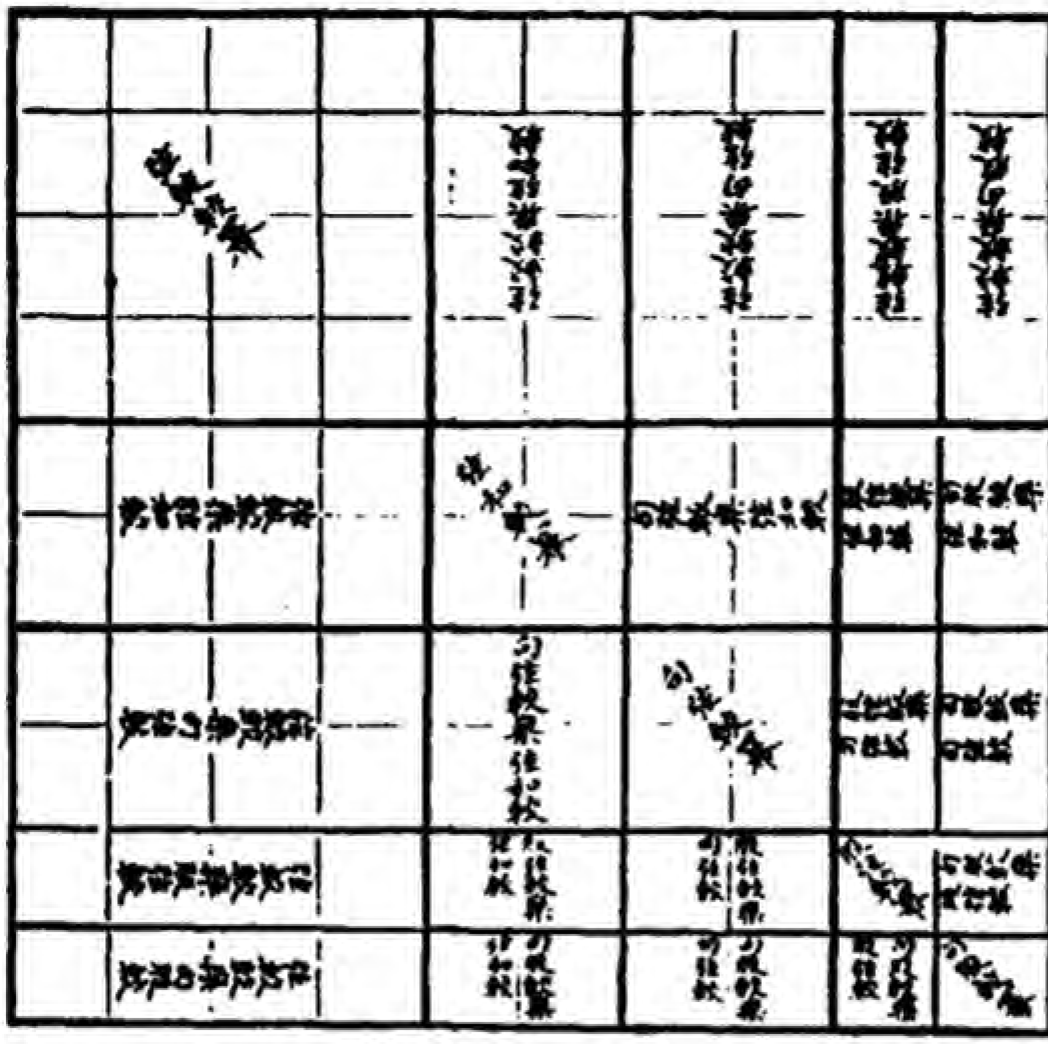
Square of (*Xian* + *He*)	(*Xian* + *He*) × (*Gu* + *Xian*)	(*Xian* + *He*) × (*Gou* + *Xian*)	(*Xian* + *He*) × (*Gou* + *Gu*)	(*Xian* + *He*) × (*Xian* + *Jiao*)
(*Xian* + *He*) × (*Gu* + *Xian*)	Square of (*Gu* + *Xian*)	(*Gu* + *Xian*) × (*Gou* + *Xian*)	(*Gu* + *Xian*) × (*Gou* + *Gu*)	(*Gu* + *Xian*) × (*Xian* + *Jiao*)
(*Xian* + *He*) × (*Gou* + *Xian*)	(*Gu* + *Xian*) × (*Gou* + *Xian*)	Square of (*Gou* + *Xian*)	(*Gou* + *Xian*) × (*Gou* + *Gu*)	(*Gou* + *Xian*) × (*Xian* + *Jiao*)
(*Xian* + *He*) × (*Gou* + *Gu*)	(*Gu* + *Xian*) × (*Gou* + *Gu*)	(*Gou* + *Xian*) × (*Gou* + *Gu*)	Square of (*Gou* + *Gu*)	(*Gou* + *Gu*) × (*Xian* + *Jiao*)
(*Xian* + *He*) × (*Xian* + *Jiao*)	(*Gu* + *Xian*) × (*Xian* + *Jiao*)	(*Gou* + *Xian*) × (*Xian* + *Jiao*)	(*Gou* + *Gu*) × (*Xian* + *Jiao*)	Square of (*Xian* + *Jiao*)

The base and the leg of a right triangle are derived from a circle and a square. By taking 1 for the diameter of a circle the circumference is then 3, and by taking 1 for the diameter of a square (a side of a square) the perimeter is 4. Putting the circumference as the base and the perimeter as the leg of a right triangle the hypotenuse of the triangle thus formed is 5. The five *he* (sums) of a right triangle are the sums of the base and the leg, the base and the hypotenuse, the leg and the hypotenuse, the hypotenuse and the *he* (the sum of the base and the leg), and the hypotenuse and the *jiao* (the difference between the leg and the base) [1] ; adding these five *he* we have 42 *bu*, the square of which equals 1764 *bu*. There are 25 sections as shown in the figure.

【 Notes 】

[1] The five *he* is as follows: $a + b$, $a + c$, $b + c$, $(a + b) + c$, and $(b - a) + c$. (G)

五較自乘演段之圖



【原文】

夫算中玄妙无过演段、如积，幽微莫越认图。其法奥妙，学者鲜能造其微。前明五和，次辨五较，自知优劣也。其五较[1]者，勾股较、勾弦较、股弦较、弦较较、弦和较。并之，得一十步。自乘，得一百步。共为二十五段。考图认之。

【注释】

[1] 五较即五差：勾与股之差 $b-a$，勾与弦之差 $c-a$，股与弦之差 $c-b$，弦与勾股差之差 $c-(b-a)$，弦与勾股和之差 $(a+b)-c$。（郭）

【今译】

算学中的玄妙，无过于演段、如积，幽微莫过于认识图形。这里的方法非常奥妙，学者很少有能体会其微旨的。先要明了五和，再辨别五较，自然就会知道优劣了。那五较就是：勾股较、勾弦较、股弦较、弦较较、弦和较。将它们相加，得到10步。再将其自乘，得到100步。总共为25段。考察图形，就理解了它们。

The Square of the Five *Jiao*

Square of (*Xian* - *jiao*)	(*Xian* - *Jiao*) × (*He* - *Xian*)	(*Xian* - *Jiao*) × (*Xiao* - *Gou*)	(*Xian* - *Jiao*) × (*Xian* - *Gu*)	(*Xian* - *Jiao*) × (*Gu* - *Gou*)
(*Xian* - *Jiao*) × (*He* - *Xian*)	Square of (*He* - *Xian*)	(*Xian* - *Gou*) × (*He* - *Xian*)	(*Xian* - *Gu*) × (*He* - *Xian*)	(*Gu* - *Gou*) × (*He* - *Xian*)
(*Xian* - *Jiao*) × (*Xian* - *Gou*)	(*Xian* - *Gou*) × (*He* - *Xian*)	Square of (*Xian* - *Gou*)	(*Xian* - *Gu*) × (*Xian* - *Gou*)	(*Gu* - *Gou*) × (*Xian* - *Gou*)
(*Xian* - *Jiao*) × (*Xian* - *Gu*)	(*Xian* - *Gu*) × (*He* - *Xian*)	(*Xian* - *Gu*) × (*Xian* - *Gou*)	Square of (*Xian* - *Gu*)	(*Gu* - *Gou*) × (*Xian* - *Gu*)
(*Xian* - *Jiao*) × (*Gu* - *Gou*)	(*Gu* - *Gou*) × (*He* - *Xian*)	(*Gu* - *Gou*) × (*Xian* - *Gou*)	(*Gu* - *Gou*) × (*Xian* - *Gu*)	Square of (*Gu* - *Gou*)

The science of form is certainly mysterious but the secret in solving a problem is to have an adequate figure. Therefore the use of figures is very important and mysterious also. In the former diagram the square of the five *he* was shown. In this is shown the square of the five *jiao*. The five *jiao* [1] are the differences between the leg and the base, the hypotenuse and the leg, the hypotenuse and the base, the *he* (the sum of the leg and the base) and the hypotenuse, the hypotenuse and the *jiao* (the difference between the leg and the base). Adding the five jiao we have 10 *bu*, the square of which equals 100 *bu*. There are 25 sections as shown in the figure.

[Notes]

[1] The five *jiao* is five differences. It includes $b - a$, $c - a$, $c - b$, $c - (b - a)$, $(a + b) - c$. (G)

四象细草假令之图

一气混元

【原文】

今有黄方[1]乘直积得二十四步。只云股弦和九步[2]，问：勾几何？

答曰：三步。

草曰：立天元一为勾，如积求之。得一百六十二个黄方乘直积式：

0太		−3888
0		0
729[3]。	以一百六十二乘元积，相消，得开方式：	729 。四乘方
−81		−81
−9		−9
1		1

开之，得勾三步。[4] 合问。[5]

【注释】

[1] 文中使用“勾”与“股”，表明图形为直角三角形，因而作为约定，诸如“在直角三角形中”这样的条件，在问题中省略。（陈）

[2] 此即：$[(a+b)-c](ab)=24$。$b+c=9$。（郭）

[3] 设勾为天元一，通过求出两个等价的多项式，如积相消求之。由 $a^2+b^2=c^2$ 得 $a^2=(c+b)(c-b)=9(c-b)$，$c-b=\frac{a^2}{9}$，而 $a+b-c=a-(c-b)=a-\frac{a^2}{9}$，故 $9a-a^2=9(a+b-c)$。又 $2b=9-\frac{a^2}{9}$，于是 $81a-a^3=18ab$。显然，$(9a-a^2)(81a-a^3)=9(a+b-c)(18ab)=162ab(a+b-c)$。原稿为算筹数字，今改为阿拉伯数字，下同。（郭）

[4] $162ab(a+b-c)=162\times24=3888$。两者如积相消，得 $a^5-9a^4-81a^3+729a^2-3888=0$。（郭）

INTRODUCTORY PROBLEMS
Yi Qi Hun Yuan (The Unitary Nebula or One Unknown Quantity)

Problem . The *huang fang* [1] multiplied by the *zhi ji* is equal to 24 *bu*; The sum of the *gu* and the *xian* is 9 *bu* [2]. Find the *gou*.

Ans. 3 *bu*.

Process. Let the element *tian* (天) be the *gou*. From the statement we have the expression

0 *tai*

0

729 [3]

−81

−9

1

for 162 times the *huang fang*, which equals 162 × 24. By cancellation we have

−3888

0

729

−81

−9

1

an expression of the fifth degree whose root, 3 *bu*[4], is the required *gou*. [5]

[5]朱世杰没有给出导出方程的方法，或开方的表述。然而，除了其是以中文符号与图形表示的之外，与现代方法是一致的。说明如下：

若 $ab(a+b-c)=24$ (1)

$b+c=9$ (2)

已知 $a^2+b^2-c^2=0$ (3)

求解 a。

(3)式除以(2)式，得 $\frac{a^2}{9}=c-b$ (4)

(4)式两端同减去 a，去分母，得

$9a-a^2=9(a+b-c)$ (5)

(2)式与(4)式相减,得 $2b=9-\frac{a^2}{9}$

或者 $18b=81-a^2$ (6)

用 a 乘以(6)式，得 $18ab=81a-a^3$ (7)

用(5)式乘以(7)式，得 $(81a-a^3)(9a-a^2)=162ab(a+b-c)$

或者 $729a^2-81a^3-9a^4+a^5=162ab(a+b-c)$ (8)

由(1)式，得 $162ab(a+b-c)=3888$

因而，(8)式变为 $a^5-9a^4-81a^3+729a^2-3888=0$

朱世杰已经给出方程(8)的左边与最后的方程，由于解方程的方法在当时已经众所周知，所以朱世杰将之省略了。如前文例7所示，在本质上，它与霍纳的方法相同。具体过程如下：

[Notes]

[1] When the words *gou* and *gu* are used, the figure is understood to be a right triangle. Therefore, the words "in a right triangle" are, as a rule, omitted in the problem. (C)

[2] That is, $[(a+b)-c](ab)=24, b+c=9$. (G)

[3] Let the element *tian* be the *gou*. By getting two polynomial expressions which are equivalence to extract and get the result. From $a^2+b^2=c^2$ have $a^2=(c+b)(c-b)=9(c-b)$, $c-b=\frac{a^2}{9}$, and $a+b-c=a-(c-b)=a-\frac{a^2}{9}$, therefore $9a-a^2=9(a+b-c)$. And $2b=9-\frac{a^2}{9}$, so $81a-a^3=18ab$. Clearly, $(9a-a^2)\cdot(81a-a^3)=9(a+b-c)(18ab)=162ab(a+b-c)$. The original text used the calculating-rods numerals.We change them into the Arabic numerals. (G)

[4] $162ab(a+b-c)=162\times 24=3888$. Extracting both two have $a^5-9a^4-81a^3+729a^2-3888=0$. (G)

[5] The method used in deriving this equation, or expression of evolution, is not given by Zhu Shijie. It is the same , however, as the modern method excepting that it is expressed in Chinese symbols and figures. For illustration:

Given $ab(a+b-c)=24$ (1)

$b+c=9$ (2)

$a^2+b^2-c^2=0$, a known law, (3)

Find a.

Dividing (3) by (2), $\frac{a^2}{9}=c-b$ (4)

Subtraction a from each member of (4) and clearing of fractions,

$9a-a^2=9(a+b-c)$ (5)

Subtraction (4) from (2), $2b=9-\frac{a^2}{9}$

or $18b=81-a^2$ (6)

Multiplying (6) by a, $18ab=81a-a^3$ (7)

Multiplying (7) by (5), $(81a-a^2)(9a-a^2)=162ab(a+b-c)$

or $729a^2-81a^3-9a^4+a^5=162ab(a+b-c)$ (8)

	3	根
	-3888	实
3 × 1296,	3888	
	0	
	0	方
3 × 432,	1296	
	1296	
	729	第一廉
3 × (-99),	-297	
	432	
	-81	第二廉
3 × (-6),	-18	
	-99	
	-9	第三廉
3 × 1,	3	
	-6	
	1	隅

水平排列

3)	1	-9	-81	+729	+	-3888
		3	-18	-297	+1296	+3888
		-6	-99	+432	+1296	0

From (1), $162ab(a+b-c)=3888$

Therefore (8) becomes, $a^5-9a^4-81a^3+729a^2-3888=0$

The first member of equation (8) and the final equation are given by Zhu Shijie, but the method of solving is omitted since it was well known to the people at that time. It is essentially Horner' s Method as shown in example 7. This particular solution is as follows:

	3	root
	−3888	*shi*
3 × 1296,	3888	
	0	
	0	*fang*
3 × 432,	1296	
	1296	
	729	1st *lian*
3 × (−99),	−297	
	432	
	−81	2nd *lian*
3 × (−6),	−18	
	−99	
	−9	3rd *lian*
3 × 1,	3	
	−6	
	1	*yu*

Arranged in the horizonal form

3)	1	−9	−81	+729	+	−3888
		3	−18	−297	+1296	+3888
		−6	−99	+432	+1296	0

朱世杰提到的“相消”是指获得同一值的两个量，且从其中一个之中减去另一个。余数的值被称作开方之式，为0。（陈）

【今译】

今有黄方乘直积得24步。只云股与弦之和为9步。问：勾为多少？

答：3步。

草：设天元一x为勾，以如积方法求其解。得到162个黄方乘直积：$x^5-9x^4-81x^3+729x^2$；以162乘原来的积，与之相消，得到开方式：$x^5-9x^4-81x^3+729x^2-3888=0$。开五次方，得到勾3步。符合所问。

"By cancellation" Zhu Shijie means obtaining two quantities of the same value and subtracting the one from the other. The value of the remainder which is called the *kai fang zhi shi* (开方之式), or the expression of evolution, is 0. (C)

两仪化元

【原文】

今有股幂减弦较较[1]，与股乘勾等。只云勾幂加弦较和，与勾乘弦同。[2]问：股几何？

答曰：四步。

草曰：立天元一为股，地元一为勾弦和。天、地配合求之，得今式：

$$\begin{array}{ccc} -2 & 0 & 太 \\ -1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}$$

，[3]求到云式：

$$\begin{array}{ccc} 2 & 0 & 太 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}$$

。[4]互隐通分，消之，内二行得式：

$$\begin{array}{c} 太 \\ 8 \\ 4 \end{array}$$

，外二行得：

$$\begin{array}{c} 0太 \\ 0 \\ 2 \\ 1 \end{array}$$

。[5]两位相消，得开方式：

$$\begin{array}{c} -8 \\ -2 \\ 1 \end{array}$$

。[6]平方开之，得股四步。合问。[7]

【注释】

[1] 在一直角三角形中，勾为a，股为b，弦为c。$c-(b-a)$为弦较较（即弦与勾股差之差）。（陈）

[2] 此即：$b^2-[c-(b-a)]=ab$。$a^2+[c+(b-a)]=ac$。（郭）

[3] 设天元一为股b，记为x；设地元一为$c+a$，记为y。此即：今式$x^3+2x^2y+2xy-2y^2-xy^2=0$。（郭）

Liang Yi Hua Yuan (The Mystery of the Two Natures or Two Unknown Quantities)

Problem . The difference between the square of the *gu* and the *xian-jiao-jiao* [1] is equal to the product of the *gu* by the *gou*; the sum of the square of the *gou* and the *xian jiao he* is equal to the product of the *gou* by the *xian*.[2] Find the *gu*.

Ans. 4 *bu*.

Process. Let the element *tian* be the *gu*, and the element *di* the sum of the *gou* and the *xian*. From the statement we have

$$\begin{array}{ccc} -2 & 0 & P \\ -1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array} \quad \text{[3], and} \quad \begin{array}{ccc} 2 & 0 & P \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \text{[4]}$$

By elimination, using the *hu yin tong fen* method (互隐通分) [method of combination] we have

$$\begin{array}{c} P \\ 8 \\ 4 \\ \end{array} \quad \text{and} \quad \begin{array}{c} P \\ 0 \text{ [5]} \\ 2 \\ 1 \end{array}$$

from the two interior and the two exterior columns respectively.

By cancellation of these columns we have

$$\begin{array}{c} -8 \\ -2 \text{ [6]} \\ 1 \end{array}$$

a quadratic expression whose root, 4 *bu*, is the required *gu*. [7]

[4] 此即：云式 $x^3+2xy-xy^2+2y^2=0$。(郭)

[5] 今式与云式相消，约去 y，得 $x^2-2y=0$。乘以 x，得 $x^3-2xy=0$。与今式相减，约去 y，得 $2x^2+4x-2y-xy=0$。此即内二行 $2x^2+4x$，或 $4x^2+8x$。外二行应为 $-2y-xy$。将 $x^2-2y=0$ 代入，化成 $x^2+\frac{1}{2}x^3$，或 $2x^2+x^3$，此即外二行。(郭)

[6] 内外二行相消，得 $x^3-2x^2-8x=0$。最后化成开方式 $x^2-2x-8=0$。(郭)

[7] 这一方法的现代数学形式如下：

若 $b^2-[c-(b-a)]=ab$ (A)

$a^2+(c+b-a)=ac$ (B)

$a^2+b^2-c^2=0$ (C)

求解 b。

令 $x=b$ (1)

及 $y=a+c$ (2)

从 (2) 式中减去 (1) 式，得 $y-x=c-b+a$ (3)

从 (1) 式中减去 (3) 式，得 $x^2-y+x=b^2-(c-b+a)$ (4)

(4) 式除以 (1) 式，得 $x-\frac{y}{x}+1=a$ (5)

从 (2) 式中减去 (5) 式，得 $y-x+\frac{y}{x}-1=c$ (6)

用 (1) 式，(5) 式及 (6) 式分别替换 (C) 式中 b，a 及 c 的值，得

$x^2+(x-\frac{y}{x}+1)^2-(y-x+\frac{y}{x}-1)^2=0$ (7)

化简，得 $x^2+2y+2xy-\frac{2y^2}{x}-y^2=0$

去分母，得 $x^3+2xy+2x^2y-2y^2-xy^2=0$ (8)

[朱世杰给出了方程 (8)]

(1) 式平方，替换 (C) 式中的 b^2，得 $x^2=b^2=c^2-a^2$ (9)

(9) 式除以 (2) 式，得 $\frac{x^2}{y}=c-a$ (10)

(2) 式加 (10) 式，得 $y+\frac{x^2}{y}=2c$ (11)

[Notes]

[1] In a right triangle whose base (*gou*) is a, leg (*gu*) b, and hypotenuse (*xian*) c, $c - (b - a)$ is called the *xian-jiao-jiao* (hypotenuse-difference-difference, that is , the difference between the hypotenuse and the difference between the leg and the base). (C)

[2] That is, $b^2 - [c - (b - a)] = ab$, $a^2 + [c + (b - a)] = ac$. (G)

[3] Let the element *tian* be the *gu* b expressed by x; let the element *di* be $c + a$, expressed by y.That is the *jin shi*: $x^3 + 2x^2y + 2xy - 2y^2 - xy^2 = 0$. (G)

[4] That is the *yun shi*: $x^3 + 2xy - xy^2 + 2y^2 = 0$. (G)

[5]Eliminating y from the expressions of *Jin* and *Yun*, Zhu Shijie had $x^2 - 2y = 0$. Multiplying the expression by x, then he had $x^3 - 2xy = 0$. Subtracting it from the *jin shi*, then by eliminating y, he had $2x^2 + 4x - 2y - xy = 0$. That is, two interior columns are $2x^2 + 4x$ or $4x^2 + 8x$, and two exterior columns are $-2y - xy$. Taking $x^2 - 2y = 0$ into $-2y - xy$, he had $x^2 + \frac{1}{2}x^3$ or $2x^2 + x^3$, which is the exterior columns. (G)

[6] By eliminating the two interior and the two exterior columns, Zhu Shijie had $x^3 - 2x^2 - 8x = 0$ and by cancellation he had $x^2 - 2x - 8 = 0$. (G)

[7] The solution in modern form is as follows:

Given $b^2 - [c - (b - a)] = ab$ (A)

$a^2 + (c + b - a) = ac$ (B)

$a^2 + b^2 - c^2 = 0$ (C)

Find b.

Let $x = b$ (1)

And $y = a + c$ (2)

Subtracting (1) from (2), $y - x = c - b + a$ (3)

Subtracting (3) from the square of (1), $x^2 - y + x = b^2 - (c - b + a)$ (4)

Dividing (4) by (1), $x - \frac{y}{x} + 1 = a$ (5)

Subtracting (5) from (2), $y - x + \frac{y}{x} - 1 = c$ (6)

从（2）式中减去（10）式，得 $y-\dfrac{x^2}{y}=2a$ （12）

用（12）式乘以（11）式，得 $y^2-\dfrac{x^4}{y^2}=4ac$ （13）

（1）加（2）式，减（12）式，得 $x+\dfrac{x^2}{y}=b-a+c$ （14）

（12）式的平方加上4倍的（14）式，得

$y^2+4x-2x^2+\dfrac{4x^2}{y}+\dfrac{x^4}{y^2}=4\,[a^2+(c+b-a)]$ （15）

将（B）式代入式（13）与（15），整理后，得

$4x-2x^2+\dfrac{4x^2}{y}+\dfrac{2x^4}{y^2}=0$

或者 $2y^2-xy^2+2xy+x^3=0$ （16）

[朱世杰亦给出了（16）式]

从（16）式中减去（8）式，化简后，得 $2y-x^2=0$ （17）

（8）式加上（17）式，化简

$2y+xy=4x+2x^2$ （18）

用（17）式乘以（18）式，化简

$x^2-2x-8=0$ （19）

[方程（19）是朱世杰给出的形式]

正根4为所求的股。

根据朱世杰所使用的方法，含有两个未知数的方程的每一项的位置如以下的图表所示。

………	*D*	*C*	*P*
………	*H*	*F*	*A*
………	*I*	*G*	*B*
	⋮	⋮	⋮

上表中*P*被用来表示太字。*P*被称为原点，或极点。因为纵列*PAB*……包含 x，x^2，x^3，…，x^n 项的系数，可以被称作 x 轴。*A*表示 x 的系数，*B*表示 x^2 的系数，等

Substituting the values of b, a, and c, given in (1), (5) and (6), in (C), we have

$$x^2 + \left(x - \frac{y}{x} + 1\right)^2 - \left(y - x + \frac{y}{x} - 1\right)^2 = 0 \qquad (7)$$

Simplifying, $x^2 + 2y + 2xy - \frac{2y^2}{x} - y^2 = 0$

Clearing of fractions, $x^3 + 2xy + 2x^2y - 2y^2 - xy^2 = 0$ (8)

[Equation (8) is given by Zhu Shijie]

Squaring (1) and substituting in (C), $x^2 = b^2 = c^2 - a^2$ (9)

Dividing (9) by (2), $\frac{x^2}{y} = c - a$ (10)

Adding (2) and (10), $y + \frac{x^2}{y} = 2c$ (11)

Subtracting (10) from (2), $y - \frac{x^2}{y} = 2a$ (12)

Multiplying (11) by (12), $y^2 - \frac{x^4}{y^2} = 4ac$ (13)

Adding (1) and (2) and subtracting (12), $x + \frac{x^2}{y} = b - a + c$ (14)

The square of (12) plus 4 times (14),

$$y^2 + 4x - 2x^2 + \frac{4x^2}{y} + \frac{x^4}{y^2} = 4\,[\,a^2 + (c + b - a)\,] \qquad (15)$$

Substituting values of (13) and (15) in (B) we have

$$4x - 2x^2 + \frac{4x^2}{y} + \frac{2x^4}{y^2} = 0$$

or $2y^2 - xy^2 + 2xy + x^3 = 0$ (16)

[Equation (16) is also given by Zhu Shijie]

Subtracting (8) from (16), and simplifying, $2y - x^2 = 0$ (17)

Adding (8) and (17), and simplifying,

$2y + xy = 4x + 2x^2$ (18)

Multiplying (18) by (17), and simplifying,

$x^2 - 2x - 8 = 0$ (19)

[Equation (19) is the form given by Zhu Shijie]

The positive root 4 is the required *gu*.

The following diagram shows the position of each term of an equation containing

等。

横排 $PCD\cdots$可以被称之为 y 轴。y，y^2，y^3，$\cdots$，$y^n\cdots$的系数被放置于该轴上。C 表示 y 的系数，D 表示 y^2 的系数，等等。

纵列 $CFG\cdots$可被称之为 y 列，y 的系数位于其上；纵列 $DHI\cdots$可被称为 y^2 列，y^2 的系数位于其上。

横排 $AFH\cdots$可被称之为 x 排，x 的系数位于其上；横排 $BGI\cdots$可被称之为 x^2 排，x^2 的系数位于其上。

从这些定义，我们容易发现，F 是 xy 项的系数，G 为 x^2y 项的系数，H 为 xy^2 项的系数，等等。

如表达式

$x^3 + 2xy + 2x^2y - 2y^2 - xy^2$

以中算形式表示为

$$\begin{array}{ccc} -2 & 0 & P \\ -1 & 2 & 0 \\ & 2 & 0 \\ & & 1 \end{array}$$

当用 x 乘以一个等于0的表达式的时候，表达式的整体形式向下移动一排；当用 x^2 乘以一个等于0的表达式的时候，表达式的整体形式向下移动两排，等等。表达式的值仍然相同，即等于0。当用 y 乘以一个等于0的表达式的时候，表达式的整体形式向左移动一列；当用 y^2 乘以一个等于0的表达式的时候，表达式的整体形式向左移动两列，等等。

当然，用除法时，表达式的形式向用乘法时所移动的反方向移动。

因此，表达式的整体形式可以向上移动，也可以向下移动，可以向左移动，也可以向右移动，并且表达式的值不变。在开方过程中，P 可被省略。

问题：解方程

two unknown quantities arranged according to the method used by Zhu Shijie.

·········	D	C	P
·········	H	F	A
·········	I	G	B
	⋮	⋮	⋮

In this discussion P is used for the Chinese character太 (*tai*). P is called the origin or the pole. The column PAB...may be called the x-axis since it contains the coefficient of $x, x^2, x^3, \cdots, x^n$

A denoting the coefficient of x, B of x^2, and so on.

The row PCD...may be called the y-axis where the coefficients of $y, y^2, y^3, \cdots, y^n$ are placed, C denoting the coefficient of y, D of y^2, and so on.

The column CFG...may be called the y column where the coefficients of y are placed; the column DHI...the y^2 column where the coefficients of y^2 are placed.

Row AFH...may be called the x row, where the coefficients of x are placed; BGI... the x^2 row where the coefficients of x^2 are placed.

From these definitions we readily see that F is the coefficient of xy, G of x^2y, H of xy^2, and so on.

Writing the expression

$x^3 + 2xy + 2x^2y - 2y^2 - xy^2$

in Chinese form we have

$$\begin{array}{ccc} -2 & 0 & P \\ -1 & 2 & 0 \\ & 2 & 0 \\ & & 1 \end{array}$$

When an expression whose value equals zero is multiplied by x, the whole form of the expression is moved one row downward; when multiplied by x^2 two rows downward, and so on. The value of the expression remains the same, that is, equal to zero. When an

$$2x^2+2x-2y-xy+\frac{x^3}{y}=0 \tag{1}$$

$$2x-x^2+\frac{x^2}{y}+\frac{x^4}{y^2}=0 \tag{2}$$

我们有

$$\begin{array}{ccc} -2 & P & 0 \\ -1 & 2 & 0 \\ 0 & 2 & 0 \\ & & \\ 0 & 0 & 1 \end{array} \quad (1) \qquad \begin{array}{ccc} P & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \quad (2)$$

去分母

$$\begin{array}{ccc} -2 & 0 & P \\ -1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array} \quad (1) \qquad \begin{array}{ccc} 2 & 0 & P \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \quad (2)$$

因为无论是用乘法，还是用除法，并不改变表达式的值，该值恒等于0，P可以被省略。从（1）式中减去（2）式，得

$$\begin{array}{cc} -4 & 0 \\ 0 & 0 \\ 0 & 2 \end{array} \quad \text{或者} \quad \begin{array}{cc} -2 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} \quad (3)$$

令（3）式与（1）式在底侧与右侧相等，从（1）式中减去（3）式，得

$$\begin{array}{cc} 2 & 0 \\ 1 & -4 \\ 0 & -2 \end{array} \quad (4)$$

并排放置（3）与（4）

$$\begin{array}{cc|cc} 2 & 0 & P & -2 \quad 0 \\ 1 & -4 & & 0 \quad 0 \\ 0 & -2 & & 0 \quad 1 \end{array} \quad (5)$$

expression whose value equals zero is multiplied by y the whole form of the expression is moved one column to the left; when multiplied by y^2 two columns to the left, and so on.

In division the form of the expression moves of course in the opposite direction, from which it moves in multiplication.

Since the whole form of the expression may be moved upward as well as downward, to the left as well as to the right, without changing its value, P may be omitted in the process of evolution.

Problem. Solve the equations,

$$2x^2 + 2x - 2y - xy + \frac{x^3}{y} = 0 \qquad (1)$$

$$2x - x^2 + \frac{x^2}{y} + \frac{x^4}{y^2} = 0 \qquad (2)$$

We have

$$\begin{matrix} -2 & P & 0 \\ -1 & 2 & 0 \\ 0 & 2 & 0 \\ & & \\ 0 & 0 & 1 \end{matrix} \quad (1) \qquad \begin{matrix} P & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \quad (2)$$

Clearing of fractions,

$$\begin{matrix} -2 & 0 & P \\ -1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix} \quad (1) \qquad \begin{matrix} 2 & 0 & P \\ -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \quad (2)$$

Since multiplication or division does not change the value of the expression, which always equals zero, P may be omitted.

Subtracting (2) from (1),

$$\begin{matrix} -4 & 0 \\ 0 & 0 \\ 0 & 2 \end{matrix} \qquad \text{or} \qquad \begin{matrix} -2 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix} \quad (3)$$

并排放置内列与外列的积

$$\begin{array}{ccc} 8 & P & 0 \\ 4 & \vert & 2 \\ 0 & \vert & 1 \end{array} \qquad (6)$$

相消，得

$$\begin{array}{c} 8 \\ 2 \\ -1 \end{array}$$

这是以 4 为根的二次表达式。

*注：在（5）式与（6）式中，P 被平放于第一排上，是为了在获得内列与外列的乘积时，其位置被明确地确定。

除了中算的计算是垂直进行，而非水平进行的之外，乘法与综合的方法相同。

例如，以 $x^2 - 7x + 5$ 乘 $x^3 + 2x^2 + x - 1$

$$\begin{array}{rrrrrr} 1 & +2 & +1 & -1 & & \\ 1 & -7 & +5 & & & \\ \hline 1 & +2 & +1 & -1 & & \\ & -7 & -14 & -7 & +7 & \\ & & 5 & +10 & +5 & -5 \\ \hline 1 & -5 & -8 & +2 & +12 & -5 \end{array}$$

中算方法为

$$\begin{array}{r|rrr|rr} -5 & & & -5 & 5 & -1 \\ 12 & & 7 & +5 & -7 & 1 \\ 2 & -1 & -7 & +10 & 1 & 2 \\ -8 & 1 & -14 & +5 & & 1 \\ -5 & 2 & -7 & & & \\ 1 & 1 & & & & \end{array}$$

（陈）

Making expressions (3) and (1) even at the bottom and at the right, and subtracting (3) from (1),

$$\begin{array}{cc} 2 & 0 \\ 1 & -4 \\ 0 & -2 \end{array} \qquad (4)$$

Placing (3) and (4) side by side,

$$\begin{array}{cc|cc} 2 & 0 & P\ -2 & 0 \\ 1 & -4 & 0 & 0 \\ 0 & -2 & 0 & 1 \end{array} \qquad (5)$$

Placing the products of the interior and exterior columns side by side,

$$\begin{array}{c|c} 8 & P\ \ 0 \\ 4 & 2 \\ 0 & 1 \end{array} \qquad (6)$$

Cancelling, we have

$$\begin{array}{c} 8 \\ 2 \\ -1 \end{array}$$

a quadratic expression with 4 for its root.

* Notice that P, in (5) and (6), is put even with the first row in order to have its position definitely decided upon when the products of the interior and exterior columns are obtained.

Remember that the multiplication is the same as in the synthetic method excepting that the Chinese compute vertically instead of horizontally.

For example, multiply $x^3 + 2x^2 + x - 1$ by $x^2 - 7x + 5$.

【今译】

今有股幂减弦较较，与股乘勾相等。只云勾幂加弦较和，与勾乘弦相等。问：股为多少？

答：4 步。

草：设天元一 x 为股，地元一 y 为勾弦和。天元与地元相配合求其解。得到今式：$x^3+2x^2y+2xy-2y^2-xy^2=0$，求到云式：$x^3+2xy-xy^2+2y^2=0$。互隐通分消元，由内二行相乘得到 $4x^2+8x$，由外二行相乘得到 $2x^2+x^3$。内外二行相消，得到开方式：$x^2-2x-8=0$。开平方，得到股四步。符合所问。

$$
\begin{array}{l}
1 + 2 + 1 - 1 \\
\underline{1 - 7 + 5 \quad\quad} \\
1 + 2 + 1 - 1 \\
\quad -7 - 14 - 7 + 7 \\
\quad\quad\quad \underline{5 + 10 + 5 - 5} \\
1 - 5 - 8 + 2 + 12 - 5
\end{array}
$$

Chinese method

-5			-5	5	-1
12		7	+5	-7	1
2	-1	-7	+10	1	2
-8	1	-14	+5		1
-5	2	-7			
1	1				

(C)

三才运元

【原文】

今有股弦较除弦和和，与直积等。只云勾弦较除弦较和与勾同。[1] 问：弦几何？

答曰：五步。

草曰：立天元一为勾，地元一为股，人元一为弦。三才相配，求得今式：$\begin{array}{cccc} 0 & -1 & \text{太} & -1 \\ & & 1 & \\ -1 & 0 & -1 & 0 \end{array}$，[2] 求得云式：$\begin{array}{ccc} -1 & \text{太} & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{array}$，[3] 求得三元之式：$\begin{array}{ccccc} 1 & 0 & \text{太} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}$。[4] 以云式剔而消之。二式皆人易天位，前得 $\begin{array}{ccc} 1 & 1 & -2\,\text{太} \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{array}$，后得 $\begin{array}{cccc} 1 & -2 & 2 & \text{太} \\ 0 & -2 & 4 & -2 \\ 0 & 0 & 1 & -2 \end{array}$。互隐通分相消，左得 $\begin{array}{cc} 7 & -6\,\text{太} \\ 3 & -7 \\ -1 & -3 \\ 0 & 1 \end{array}$，右得 $\begin{array}{cc} 13 & -14\,\text{太} \\ 11 & -13 \\ 5 & -15 \\ -2 & -5 \\ 0 & 2 \end{array}$。内二行得 $\begin{array}{c} -78 \\ -157 \\ -146 \\ -43 \\ 10 \\ 11 \\ -2 \end{array}$，外二行得 $\begin{array}{c} -98 \\ -133 \\ -130 \\ -67 \\ 14 \\ 11 \\ -2 \end{array}$。内外相消，四约之，得开方式 $\begin{array}{c} -5 \\ 6 \\ 4 \\ -6 \\ 1 \end{array}$。三乘方开之，得弦五步。[5] 合问。

San Cai Yun Yuan (The Evolution of the Three Talents or Three Unknown Quantities)

Problem. The quotient of the *xian he he* and the difference between the *xian* and the *gu* is equal to *zhi ji*; the quotient of the *xian jiao he* and the difference between the *xian* and the *gou* is equal to the *gou*. [1] Find the *xian*.

Ans. 5 *bu*.

Process. Let the element *tian* be the *gou*, the element *di* the *gu*, and the element *ren* the *xian*. Using these three talents (elements) we have

$$\begin{array}{cccc} 0 & -1 & P & -1 \\ & 1 & & \\ -1 & 0 & -1 & 0 \end{array}, [2] \qquad \begin{array}{ccc} -1 & P & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{array}, [3] \qquad \begin{array}{ccccc} 1 & 0 & P & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} [4]$$

Eliminating and changing the positions of *tian* and *ren* we have

$$\begin{array}{ccc} 1 & 1 & -2P \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{array} \quad \text{and} \quad \begin{array}{cccc} 1 & -2 & 2 & P \\ 0 & -2 & 4 & -2 \\ 0 & 0 & 1 & -2 \end{array}$$

By composition,

$$\begin{array}{cc|cc} 7 & -6 & 13 & -14 \\ 3 & -7 & 11 & -13 \\ -1 & -3 & 5 & -15 \\ 0 & 1 & -2 & -5 \\ & & 0 & 2 \end{array}$$

(with P printed above the dividing line)

The products of the interior and exterior columns are

【注释】

[1]此即：$\frac{(a+b)+c}{c-b}=ab$。$\frac{c+(b-a)}{c-a}=a$。(郭)

[2]设天元一x为勾，地元一y为股，人元一z为弦，将注[1]第一式展开，即是今式：$-x-y-z+xyz-xy^2=0$。(郭)

[3]将注[1]第二式展开，即是云式：$x-y-z-x^2+xz=0$。(郭)

[4]由勾股术得三元式：$x^2+y^2-z^2=0$。(郭)

[5]以云式减今式，除以x，将云式和三元式变成y的函数代入，在三元术中，还要将人元摆到天元上，得到前式：$x^2+x-x^2z+xz-z+xz^2-2z^2-2=0$；

将云式的关于y的函数代入三元式，也将人元摆到天元上，得到后式：

$x^3-2x^2+2x-2x^2z+4xz-2z+xz^2-2z^2=0$。

互隐通分相消，得到左式：

$(-z^2+3z+7)x+(z^3-3z^2-7z-6)=0$，

及右式：

$(-2z^3+5z^2+11z+13)x+(2z^4-5z^3-15z^2-13z-14)=0$。

内二行相乘得

$-2z^6+11z^5+10z^4-43z^3-146z^2-157z-78$，

外二行相乘得

$-2z^6+11z^5+14z^4-67z^3-130z^2-133z-98$。

内外相消，以4约之，得到开方式

$z^4-6z^3+4z^2+6z-5=0$。(郭)

【今译】

今有以股弦较除弦和和，等于长方形的面积。只云以勾弦较除弦较和，等于勾。问：弦为多少？

答：5步。

-78	P	-99
-157		-133
-146		-130
-43		-67
10		14
11		11
-2		-2

-5

6

After cancellation and dividing by 4, we have 4 , an expression of the

-6

1

fourth degree whose root, 5, [5] is the required *gou*.

[Notes]

[1] That is, $\frac{(a+b)+c}{c-b} = ab$, $\frac{c+(b-a)}{c-a} a$. (G)

[2] Let x be *tian*, y *di*, z *ren*. Using them to replace a, b, c in the first expression of last footnote, then get the *jin shi*: $-x - y - z + xyz - xy^2 = 0$. (G)

[3] Let x, y, and z replace a, b, c in the second expression of the footnote before the last, then get the Yun shi: $x - y - z - x^2 + xz = 0$. (G)

[4] A *san yuan shi* is gotten by the methods of solving a right triangle: $x^2 + y^2 - z^2 = 0$. (G)

[5] Subtracting the *jin shi* from the *yun shi*, and dividing by x, making the *yun shi* and the *san yuan shi* become the function of y, in *san yuan shu* putting the element *ren* on the element *tian* as well, we have the *qian shi*:

$$x^2 + x - x^2z + xz - z + xz^2 - 2z^2 - 2 = 0;$$

草：设天元一x为勾，地元一y为股，人元一z为弦，天元、地元与人元三者相配合以求其解。得到今式：$-x-y-z+xyz-xy^2=0$，求得云式：$x-y-z-x^2+xz=0$，求得三元式：$x^2+y^2-z^2=0$。以云式对今式和三元式剔而消之，二式皆将人元换到天元的位置上，得到前式：$x^2+x-x^2z+xz-z+xz^2-2z^2-2=0$，得到后式：$x^3-2x^2+2x-2x^2z+4xz-2z+xz^2-2z^2=0$。互隐通分，相消，得到左式：

$$(-z^2+3z+7)x+(z^3-3z^2-7z-6)=0,$$

右式：

$$(-2z^3+5z^2+11z+13)x+(2z^4-5z^3-15z^2-13z-14)=0。$$

内二行相乘得$-2z^6+11z^5+10z^4-43z^3-146z^2-157z-78$，

外二行相乘得$-2z^6+11z^5+14z^4-67z^3-130z^2-133z-98$。

内外二行相消，以四约之，得到开方式：

$$z^4-6z^3+4z^2+6z-5=0。$$

开四次方，得到弦5步。符合所问。

Substituting the function concerning y conversed by the *yun shi* for y in the *san yuan shi*, then also putting the element *ren* on the element *tian*, we have the *hou shi*:

$x^3 - 2x^2 + 2x - 2x^2z + 4xz - 2z + xz^2 - 2z^2 = 0.$

By cancellation, we have the *zuo shi*:

$(-z^2 + 3z + 7)x + (z^3 - 3z^2 - 7z - 6) = 0,$

And *you shi*:

$(-2z^3 + 5z^2 + 11z + 13)x + (2z^4 - 5z^3 - 15z^2 - 13z - 14) = 0.$

Multiplying the two interior columns, we get:

$-2z^6 + 11z^5 + 10z^4 - 43z^3 - 146z^2 - 157z - 78,$

Multiplying the exterior two columns, we get:

$-2z^6 + 11z^5 + 14z^4 - 67z^3 - 130z^2 - 133z - 98.$

After cancellation and dividing by 4, we have $z^4 - 6z^3 + 4z^2 + 6z - 5 = 0.$ (G)

四象会元

【原文】

今有股乘五较，与弦幂加勾乘弦等。只云勾除五和，与股幂减勾弦较同。[1] 问：黄方带勾股弦共几何？

答曰：一十四步。

草曰：立天元一为勾，地元一为股，人元一为弦，物元一为开数。四象和会求之，求得今式 $\begin{array}{ccc} -2 & 太 & 1 \\ 0 & 1 & 0 \end{array}$（1），[2]求得云式 $\begin{array}{cccc} 0 & 4 & 太 & 4 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{array}$（2），[3]求得三元之式 $\begin{array}{ccccc} 1 & 0 & 太 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}$（3），[4]求得物元之式 $\begin{array}{ccc} 0 & -1 & 0 \\ 2 & 太 & 0 \\ 0 & 2 & 0 \end{array}$（4）。[5]

四式和会，消而剔之，皆物易天位，得前式 $\begin{array}{cccc} 2 & -22 & 28 & 太 \\ 0 & -1 & 10 & -2 \\ 0 & 0 & 0 & -1 \end{array}$（5），后式 $\begin{array}{cc} -7 & 太 \\ 0 & 2 \end{array}$（6），便为左行。以左行消前式 $\begin{array}{cc} 0 & 294 \\ 8 & 3 \\ 0 & -4 \end{array}$，便为右行。内二行得式 $\begin{array}{c} 0 \\ 0 \\ 16 \end{array}$，其外二行得式 $\begin{array}{c} -2058 \\ -21 \\ 28 \end{array}$。内外二行相消，三约，得开方式 $\begin{array}{c} -686 \\ -7 \\ 4 \end{array}$。平方开之，得一十四步。[6] 合问。[7]

【注释】

[1] 记五和之和为 $A=(a+b)+(a+c)+(b+c)+[c+(a+b)]$

Si Xiang Hui Yuan (Simultaneousness of the Four Phenomena or Simultaneous Equations in Four Unknowns)

Problem. The product of the *gu* by the five *jiao* is equal to the sum of the square of the *xian* and the product of the *gou* by the *xian*; The quotient of the five *he* and the *gou* is equal to the difference between the square of the *gu* and the difference between the *xian* and *gou*. [1] Find the sum of the *huang fang*, *gou*, *gu* and *xian*.

Ans. 14 *bu*.

Process. Let the element *tian* be the *gou*, the element *di* the *gu*, the element *ren* the *xian*, and the element *wu* the sum of the *huang fang*, *gou*, *gu*, and *xian*. These four phenomena meeting we obtain the following expressions

$$\begin{array}{ccc} -2 & P & 1 \\ 0 & 1 & 0 \end{array}\ (1),^{[2]}\quad \begin{array}{cccc} 0 & 4 & P & 4 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{array}\ (2),^{[3]}\quad \begin{array}{ccccc} 1 & 0 & P & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\ (3)^{[4]}$$

$$\begin{array}{ccc} 0 & -1 & 0 \\ 2 & P & 0 \\ 0 & 2 & 0 \end{array}\ (4)^{[5]}$$

By eliminating and interchanging the positions of *tian* and *wu*,

$$\begin{array}{cccc} 2 & -22 & 28 & P \\ 0 & -1 & 10 & -2 \\ 0 & 0 & 0 & -1 \end{array}\ (5), \text{ and } \begin{array}{cc} -7 & P \\ 0 & 2 \end{array}\ (6)$$

From these expressions we obtain also

$+[c+(b-a)]$，五较之和为 $B=(b-a)+(c-a)+(c-b)+[c-(b-a)]+[(a+b)-c]$，此即：$bB=c^2+ac$，$\frac{A}{a}=b^2-(c-a)$。（郭）

[2] 设天元一 x 为勾，地元一 y 为股，人元一 z 为弦，物元一 u 为所求数 $[(a+b)-c]+(a+b+c)$。五较之和 $B=2c$，因此 $2bc=c^2+ac$，求得今式：$x-2y+z=0$。（郭）

[3] 五和之和 $A=2a+4b+4c$，因此 $2a+4b+4c=a[b^2-(c-a)]$，求得云式：$-x^2+2x-xy^2+4y+xz+4z=0$。（郭）

[4] 由勾股术求得三元式：$x^2+y^2-z^2=0$。（郭）

[5] 黄方与勾、股、弦三者之和为 $u=2a+2b$，其为物元，于是求得物元式：$2x+2y-u=0$。（郭）

[6] 将今式加倍，与物元式相消，得到上位 $-6y+2z+u=0$。剔云式与三元式相消，又以物元式消之，得到中位 $-2y^3-2y^2-4y+2yz+y^2u+2z^2-8z-zu-2u=0$。剔三元式与物元式相消，得到下位 $8y^2-4yu-4z^2+u^2=0$。剔上位、中位相消，以4约之，$2y^3-22y^2+28y-y^2u+10yu-u^2-2u=0$。在四元消法中，还要将物元摆到天元上，就是前式：$2y^3-22y^2+28y-y^2x+10yx-x^2-2x=0$。前式中 y^2 的系数原文误作：-8，今依田淼校正。剔上位、下位相消，以16约之，得到 $-7y+2u=0$，也将物元摆到天元上，就是后式。后式即是左式。以左式消前式得到右式：

$8u^2y-4u^2+3u+294=0$

内二行相乘得

$2(8u^2)=16u^2$，

外二行相乘得

$-7(-4u^2+3u+294)=28u^2-21u-2058$。

内外二行相消，以3约之，得到开方式：

0	294
8	3
0	−4

Which is placed on the right (of the bar) and (6) on the left. The product of the interior and exterior columns are

0		−2058
0	and	−21
16		28

Subtracting the one from the other and dividing by 3, we have

$$\begin{matrix} -686 \\ -7 \\ 4 \end{matrix},$$

a quadratic expression whose root, 14, [6] is the required number. [7]

【Notes】

[1] Use $A = (a + b) + (a + c) + (b + c) + [c + (a + b)] + [c + (b - a)]$ for the sum of the five *he*, $B = (b - a) + (c - a) + (c - b) + [c - (b - a)] + [(a + b) - c]$ for the sum of the five *jiao*. That is, $bB = c^2 + ac$, $\frac{A}{a} = b^2 - (c - a)$. (G)

[2] Use the element tian x for the *gou*, the element *di* y for the *gu*, the element *ren* z for the *xian*, and the element *wu* u for the required number which is $[(a + b) - c] + (a + b + c)$. The sum of five *jiao* is $A = 2c$. So $2bc = c^2 + ac$, we have *Jin shi*: $x - 2y + z = 0$. (G)

[3] The sum of five *he* is $B = 2a + 4b + 4c$. Therefore, $2a + 4b + 4c = a[b^2 - (c - a)]$. We have the *yun shi*: $-x^2 + 2x - xy^2 + 4y + xz + 4z = 0$. (G)

[4] A *san yuan shi* is gotten by the methods of solving a right triangle: $x^2 + y^2 - z^2 = 0$. (G)

$4u^2 - 7u - 686 = 0$。（郭）

[7]本书正文第70页所述前6个算式中的式（1）被朱世杰称为今式。今是题设的第一个字。通过题设可以得到（1）式；因为题设的第二部分以“只云”开始，所以（2）式被称作云式；因为（3）式包含三个元：天、地、人，所以被称作三元式；因为（4）式包含物元，所以它被称为物元式；（5）式被称为前式；（6）式被称为后式。

为了表述方便，根据现代方法，这些表达式以数码编列。解题方法如下所示：

令x为勾，y为股，z为弦，w为黄方、勾、股，以及弦的和。五较为$y-x$，$z-x$，$z-y$，$x+y-z$，以及$z-(y-x)$。五较的和为$2z$。五和为$x+y$，$x+z$，$y+z$，$x+y+z$，以及$z+(y-x)$。五和之和为$2x+4y+4z$。

由题设的第一部分，可得$2yz = z^2 + xz$

或者$z^2 + xz - 2yz = 0$, (1)

由题设的第二部分，可得 $\dfrac{2x+4y+4z}{x} = y^2 - z + x$

或者$2x + 4y + 4z = xy^2 - xz + x^2$

或者$2x + 4y + 4z - xy^2 + xz - x^2 = 0$ (2)

且$x^2 + y^2 = z^2$或者$x^2 + y^2 - z^2 = 0$ (3)

及$w = x + y - z + x + y + z = 2x + 2y$

或$2x + 2y - w = 0$ (4)

将以上各式转换为中算形式为

$$\begin{matrix} -2 & P & 1 \\ 0 & 1 & 0 \end{matrix} \quad (1), \qquad \begin{matrix} 0 & 4 & P & 4 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{matrix} \quad (2),$$

$$\begin{matrix} 1 & 0 & P & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix} \quad (3), \qquad \begin{matrix} 0 & -1 & 0 \\ 2 & P & 0 \\ 0 & 2 & 0 \end{matrix} \quad (4)$$

如朱世杰所给出的那样。

[5] The sum of the *huang fang*, *gou*, *gu* and *xian* is $u = 2a + 2b$, that is the element *wu*. So we have the *wu yuan shi*: $2x + 2y - u = 0$. (G)

[6] Multiplying the *jin shi* by 2, and eliminating with *wu yuan shi*, we have the top position, $-6y + 2z + u = 0$. Eliminating the *yun shi* and by cancellation with the san *yuan shi*, then with the *wu yuan shi*, we have the centre position, $-2y^3 - 2y^2 - 4y + 2yz + y^2u + 2z^2 - 8z - zu - 2u = 0$. Eliminating the *san yuan shi* and by cancellation with the *wu yuan shi*, we have the bottom position, $8y^2 - 4yu - 4z^2 + u^2 = 0$. Eliminating the top position and by cancellation with the center position, then dividing the result by 4, we have $2y^3 - 22y^2 + 28y - y^2u + 10yu - u^2 - 2u = 0$. In four elements' elimination, the element *wu* need to be put on the element *tian*. So we have the *qian shi*: $2y^3 - 22y^2 + 28y - y^2x + 10yx - x^2 - 2x = 0$. In the *qian shi* the coefficent of y^2 was mistaken for -8. Now we correct it according to Tian Miao's collation. Eliminating the top position and by cancellation with the bottom position, then dividing the result by 16, we have $-7y + 2u = 0$. Putting the element *wu* on the element *tian* too, we have the *hou shi*: $-7y + 2x = 0$. The *hou shi* is the *zuo shi*. Eliminating the *qian shi* with the *zuo shi*, we have the *you shi*: $8u^2y - 4u^2 + 3u + 294 = 0$.

Multiplying the two interior columns, we have $2(8u^2) = 16u^2$.

Multiplying the two exterior columns, we have $-7(-4u^2 + 3u + 294) = 28u^2 - 21u - 2058$.

Eliminating the two interior and the two euterior columns, then dividing the result by 3, we have the quadratic expression $4u^2 - 7u - 686 = 0$. (G)

[7] In the expressions of the text in P.71, expression(1)is called, by Zhu Shijie, *jin shi* (今式). *Jin* (今, now) is the first character used in the beginning of the statement from which the first expression is obtained; expression (2) is called *yun shi* (云式) because the second part of the statement begins with the two words *zhi yun* (只云) or "and it is said"; expression (3) is called *san yuan shi* (三元式), or three elements expression, be-

[用 x 表示天，y 表示地，z 表示人，w 表示物]

因为我们要得到关于 w 的方程，所以交换（4）式中 x 与 w 的位置

$$\begin{array}{cc} 0 & 2 \\ 2 & P \\ 0 & -1 \end{array} \qquad (4a)$$

将此式分为两个部分

$$\begin{array}{cc} 0 & 2 \\ 2 & P \end{array} \quad 与 \quad \begin{array}{c} P \\ -1 \end{array}$$

两个部分均平方，然后从一个部分中减去另一个部分，得

$$\begin{array}{ccc} 0 & 0 & -4 \\ 0 & -8 & 0 \\ -4 & 0 & P \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \qquad (5a)$$

以（2）式右列乘（1）式，得

$$\begin{array}{ccc} -8 & P & 4 \\ -2 & 4 & 1 \\ 0 & 1 & 0 \end{array} \qquad (6)$$

从（6）式中减去（2）式

$$\begin{array}{ccc} 0 & -12 & P \\ 1 & -2 & 2 \\ 0 & 0 & 2 \end{array}$$

交换此表达式中 x 与 w 的位置

$$\begin{array}{ccc} 0 & 0 & 2 \\ 1 & -2 & 2 \end{array} \qquad (7a)$$

cause it contains the three elements, *tian*, *di* and *ren*. Expression (4) is called *wu yuan shi* (物元式) because it contains the element *wu*. Expression (5) is called *qian shi* (前式), and expression (6) is called *hou shi* (后式) or former and latter expressions.

In these discussions the expressions are numbered according to the modern method merely for convenience. The process of solution is as follows.

Let x be the base, y the leg, z the hypotenuse, and w the sum of the *huang fang*, base, leg, and hypotenuse. The five *jiao* are $y - x$, $z - x$, $z - y$, $x + y - z$, and $z - (y - x)$, the sum of which is equal to $2z$. The five *he* are $x + y$, $x + z$, $y + z$, $x + y + z$, and $z + (y - x)$, the sum of which is equal to $2x + 4y + 4z$.

From the first statement we have $2yz = z^2 + xz$

or $z^2 + xz - 2yz = 0$, (1)

From the second statement we have $\dfrac{2x + 4y + 4z}{x} = y^2 - z + x$

or $2x + 4y + 4z = xy^2 - xz + x^2$

or $2x + 4y + 4z - xy^2 + xz - x^2 = 0$ (2)

And $x^2 + y^2 = z^2$ or $x^2 + y^2 - z^2 = 0$ (3)

And $w = x + y - z + x + y + z = 2x + 2y$

or $2x + 2y - w = 0$ (4)

Putting these into the Chinese form we have

$$\begin{array}{ccc} -2 & P & 1 \\ 0 & 1 & 0 \end{array} \quad (1), \qquad \begin{array}{cccc} 0 & 4 & P & 4 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{array} \quad (2),$$

$$\begin{array}{ccccc} 1 & 0 & P & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \quad (3), \qquad \begin{array}{ccc} 0 & -1 & 0 \\ 2 & P & 0 \\ 0 & 2 & 0 \end{array} \quad (4)$$

0　-12　*P*

以2乘（7a）式，然后加上（5a）式，得

2	-12	4	
-4	-24	*P*	（8a）
0	0	0	
0	0	1	

以2乘（4a）式，然后从（8a）中将结果减去，得

2	-12	0	
-4	-28	*P*	（9a）
0	0	2	
0	0	1	

令（4a）式的右列与（9a）式的左列相等，然后相减，得

0	0	-12	0	
-2	-4	-28	*P*	（10a）
0	1	0	2	
0	0	0	1	

以6乘（4a）式，然后加上（10a），得

-2	8	-28	*P*	
0	1	-6	2	（11a）
0	0	0	1	

由 z 自身，将（1）式分成两个部分

-2　*P*　与　*P*　1

0　1

两个部分均平方，然后从一个部分中减去另一个部分，得

4　0　*P*　0　-1

as given by Zhu Shijie.

[Use x for *tian*, y for *di*, z for *ren*, and w for *wu*.]

Since we want to form the equation in terms of w, interchange the position of x and w in expression (4),

$$\begin{array}{cc} 0 & 2 \\ 2 & P \\ 0 & -1 \end{array} \qquad (4a)$$

Separate this expression into two parts,

$$\begin{array}{cc} 0 & 2 \\ 2 & P \end{array} \quad \text{and} \quad \begin{array}{c} P \\ -1 \end{array}$$

Square both parts and subtract the one from the other,

$$\begin{array}{ccc} 0 & 0 & -4 \\ 0 & -8 & 0 \\ -4 & 0 & P \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \qquad (5a)$$

Multiply (1) by the right column of (2),

$$\begin{array}{ccc} -8 & P & 4 \\ -2 & 4 & 1 \\ 0 & 1 & 0 \end{array} \qquad (6)$$

Subtract (2) from (6),

$$\begin{array}{ccc} 0 & -12 & P \\ 1 & -2 & 2 \\ 0 & 0 & 2 \end{array}$$

Interchange the position of x and w of this expression,

0	−4	0	0	0	(12)
0	0	1	0	0	

从（3）式中减去（12）式

−3	P
0	4

交换 x 与 w 的位置

0	4	
−3	P	(13a)

以2乘（4a）式，然后从（13a）式中减去这个结果

−7	P	
0	2	(14a)

从（11a）式中减去（14a）式，得

−2	8	−21	P	
0	1	−6	0	(15a)
0	0	0	1	

以（14a）的右列乘（15a）式

−4	16	−42	P	
0	2	−12	0	(16a)
0	0	0	2	

令（16a）式与（14a）式的下排相等，然后相减

−4	16	−42	
0	2	5	(17a)

以（17a）的右列乘（14a）式

294	P	
35	−84	(18a)

$$\begin{matrix} 0 & 0 & 2 \\ 1 & -2 & 2 \\ 0 & -12 & P \end{matrix} \qquad (7a)$$

Multiply (7a) by 2, then add to (5a),

$$\begin{matrix} 2 & -12 & 4 \\ -4 & -24 & P \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad (8a)$$

Multiply (4a) by 2, then subtract from (8a),

$$\begin{matrix} 2 & -12 & 0 \\ -4 & -28 & P \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{matrix} \qquad (9a)$$

Make the right column of (4a) even with the left column of (9a), and subtract,

$$\begin{matrix} 0 & 0 & -12 & 0 \\ -2 & -4 & -28 & P \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad (10a)$$

Multiply (4a) by 6 then add to (10a),

$$\begin{matrix} -2 & 8 & -28 & P \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad (11a)$$

Separate (1) into two parts by setting z by itself,

$$\begin{matrix} -2 & P \\ 0 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} P & 1 \end{matrix}$$

$$\begin{array}{cc} 0 & -10 \end{array}$$

以（14a）的右列乘（17a）式

$$\begin{array}{ccc} -8 & 32 & -84 \\ 0 & 4 & -10 \end{array} \qquad (19a)$$

从（18a）式中减去（19a）式

$$\begin{array}{cc} 0 & 294 \\ 8 & 3 \\ 0 & -4 \end{array} \qquad (20a)$$

令（14a）式与（20a）式并排

$$\begin{array}{cc|cc} -7 & 0 \quad P & 0 & 294 \\ 0 & 2 & 8 & 3 \\ 0 & 0 & 0 & -4 \end{array}$$

内外列的积为

$$\begin{array}{c|c} 0 \quad P & -2058 \\ 0 & -21 \\ 16 & 28 \end{array}$$

相减，然后用 3 除所得的差，从而我们得到

$$\begin{array}{c} -686 \\ -7 \\ 4 \end{array}$$

这个二次表达式的根为 14，即为所求。（陈）

【今译】

今有股乘五较，与弦幂加上勾乘弦相等。只云：以勾除五和，与股幂减去勾弦较相等。问：黄方加上勾、股、弦共得多少？

Square both, then subtract one from the other,

4	0	*P*	0	−1	
0	−4	0	0	0	(12)
0	0	1	0	0	

Subtract (12) from (3),

−3	*P*
0	4

Interchange the position of x and w,

0	4	
−3	*P*	(13a)

Multiply (4a) by 2, then subtract from (13a),

−7	*P*	
0	2	(14a)

Subtract (14a) from (11a),

−2	8	−21	*P*	
0	1	−6	0	(15a)
0	0	0	1	

Multiply (15a) by the right column of (14a),

−4	16	−42	*P*	
0	2	−12	0	(16a)
0	0	0	2	

Make the lower rows of (16a) and (14a) even, then subtract,

−4	16	−42	
0	2	5	(17a)

Multiply (14a) by the right column of (17a),

答：14步。

草：设天元一 x 为勾，地元一 y 为股，人元一 z 为弦，物元一 u 为所求数，天元、地元、人元、物元四者相和会以求其解。得到今式：$x-2y+z=0$，求得云式：$-x^2+2x-xy^2+4y+xz+4z=0$，求得三元式：$x^2+y^2-z^2=0$，求得物元式：$2x+2y-u=0$。和会消而剔之，皆将物元换到天元的位置上，得到前式：$2y^3-22y^2+28y-y^2x+10yx-x^2-2x=0$；又得到后式：$-7y+2x=0$，就是左式。以左式去消前式，得到 $8x^2y-4x^2+3x+294=0$，就是右式。左式与右式的内二行相乘，得到 $16x^2$；外二行相乘得到 $28x^2-21x-2058$。此二多项式相消，以3约简，得到开方式：$4x^2-7x-686=0$。开平方，得到14步。符合所问。

$$\begin{array}{cc} 294 & P \\ 35 & -84 \\ 0 & -10 \end{array} \qquad (18a)$$

Multiply (17a) by the right column of (14a),

$$\begin{array}{ccc} -8 & 32 & -84 \\ 0 & 4 & -10 \end{array} \qquad (19a)$$

Subtract (19a) from (18a),

$$\begin{array}{cc} 0 & 294 \\ 8 & 3 \\ 0 & -4 \end{array} \qquad (20a)$$

Put (14a) and (20a) side by side,

$$\begin{array}{cc|cc} -7 & 0 \quad P & 0 & 294 \\ 0 & 2 & 8 & 3 \\ 0 & 0 & 0 & -4 \end{array}$$

The product of the interior and exterior columns are

$$\begin{array}{c|c} 0 \quad P & -2058 \\ 0 & -21 \\ 16 & 28 \end{array}$$

Subtract, then divide the remainder by 3, and we have

$$\begin{array}{c} -686 \\ -7 \\ 4 \end{array}$$

a quadratic expression whose root, 14, is the required number. (C)

四元玉鉴　卷上

直段求源　一十八问

1.

【原文】

今有弦和和乘三相和，加弦幂，共得一百六十九步。[1] 只云弦较较乘弦较和，减股弦和乘股弦较，余一十五步。[2] 问：勾几何？

答曰：三步。

术曰：立天元一为勾，如积求之。得二十五万三千一百二十五为正实。八十一万八千一百为益上廉，二十七万八千九百二十六为从三廉，二万二千八百六十八为益五廉，一百八十一为从隅，七乘方开之，[3] 得勾。合问。

【注释】

[1] 记勾股形的勾、股、弦分别为 a, b, c，此即：$[c+(a+b)](a+b+c)+c^2=169$。（郭）

[2] 此即：$[c-(b-a)][c+(b-a)]-(c+b)(c-b)=15$。（郭）

[3] 开方式的现代形式为：

$181x^8-22868x^6+278926x^4-818100x^2+253125=0$。（陈）

【今译】

今有弦和和与勾、股、弦三者之和相乘，加弦幂，共得169步。只云弦较较与弦较和的乘积，减去股弦和与股弦较的乘积，余15步。问：勾为多少？

答：3步。

术：设天元一为勾，以如积方法求其解。得到253125为常数项，−818100为二次项系数，278926为四次项系数，−22868为六次项系数，181为八次项系数，开八次方，便得到勾。符合所问。

BOOK I
Zhi Duan Qiu Yuan (Problems on Right Triangles and Rectangles)
18 Problems

1. The product of the *xian he he* by the sum of the three sides plus the square of the xian is equal to 169 *bu*[1], and the product of the *xian jiao jiao* by the *xian jiao he* minus the poduct of the *gu xian he* by the *gu xian jiao* is equal to 15 *bu*.[2] Find the *gou*.

Ans. 3 *bu*.

Process. Let the element *tian* be the *gou*. From the statement we have 253125 for the positive *shi*, 818100 for the negative first *lian*, 278926 for the positive third *lian*, 22868 for the negative fifth *lian*, and 181 for the positive *yu*, an expression[3] of the eighth degree whose root is the required *gou*.

[Notes]

[1] Use a for the *gou*, b for the *gu*, and c for the *xian*, that is $[c+(a+b)](a+b+c)+c^2=169$. (G)

[2] That is, $[c-(b-a)][c+(b-a)]-(c+b)(c-b)=15$. (G)

[3] The expression in modern form is the equation: $181x^8-22868x^6+278926x^4-818100x^2+253125=0$. (C)

2.

【原文】

今有勾弦和乘股弦和，减勾弦较乘股弦较，余七十步。[1] 只云弦和和乘弦较和，得七十二步。[2] 问：股几何？

答曰：四步。

术曰：立天元一为股，如积求之。得一百六十七万九千六百一十六为正实，一十八万六千六百二十四为益上廉，六千四百七十九为从三廉，七十为益五廉，一为益隅，七乘方开之，[3] 得股。合问。

【注释】

[1] 此即：$(c+a)(c+b)-(c-a)(c-b)=70$。（郭）

[2] 此即：$[c+(b+a)][c+(b-a)]=72$。（郭）

[3] 开方式的现代形式为：$-x^8-70x^6+6479x^4-186624x^2+1679616=0$。（陈）

【今译】

今有勾弦和与股弦和的乘积，减去勾弦较与股弦较的乘积，余70步。只云弦和和乘弦较和得72步。问：股为多少？

答：4步。

术：设天元一为股，以如积方法求其解。得到1679616为常数项，−186624为二次项系数，6479为四次项系数，−70为六次项系数，−1为八次项系数，开八次方，便得到股。符合所问。

3.

【原文】

今有弦和较乘弦和和，加勾弦较乘勾弦和，得四十步。[1] 只云勾幂与股弦和等[2]。问：弦几何？

答曰：五步。

术曰：立天元一为弦，如积求之。得一千五百二十五为正实，一百四十为从方，五十四为益上廉，一十二为益下廉，一为正隅，三乘方开之，[3] 得弦。合问。

2. The product of the *gou xian he* by the *gu xian he* minus the product of the *gou xian jiao* by the *gu xian jiao* is equal to 70 *bu*[1], and the product of the *xian he he* by the *xian jiao he* is equal to 72 *bu*.[2] Find the gu.

Ans. 4 *bu*.

Process. Let the element *tian* be the *gu*. From the statement we have 1679616 for the positive *shi*, 186624 for the negative first *lian*, 6479 for the positive third *lian*, 70 for the negative fifth *lian*, and 1 for the negative *yu*, an expression[3] of the eighth degree whose root is the required *gu*.

[Notes]

[1] That is, $(c+a)(c+b)-(c-a)(c-b)=70$. (G)

[2] That is, $[c+(b+a)][c+(b-a)]=72$. (G)

[3] The expression in modern form is the equation: $-x^8-70x^6+6479x^4-186624x^2+1679616=0$. (C)

3. The product of the *xian he jiao* by the *xian he he* plus the product of the *gou xian jiao* by the *gou xian he* is equal to 40 *bu*[1], and the square of the *gou* is equal to the *gu xian he*[2]. Find the *xian*.

Ans. 5 *bu*.

Process. Let the element *tian* be the *xian*. From the statement we have 1525 for the positive *shi*, 140 for the positive *fang*, 54 for the negative *lian*,

【注释】

[1] 此即：$[(a+b)-c][(a+b)+c]+(c-a)(c+a)=40$。（郭）

[2] 此即：$a^2=c+b$。（郭）

[3] 开方式的现代形式为：$x^4-12x^3-54x^2+140x+1525=0$。（陈）

【今译】

今有弦和较与弦和和的乘积，加勾弦较与勾弦和的乘积，得40步。只云勾幂与股弦和相等。问：弦为多少？

答：5步。

术：设天元一为弦，以如积方法求其解。得到1525为常数项，140为一次项系数，−54为二次项系数，−12为三次项系数，1为最高次项系数，开四次方，便得到弦。符合所问。

4.

【原文】

今有积减弦和较，余一十步。[1] 只云勾股和七步[2]。问：黄方几何？

答曰：二步。

术曰：立天元一为黄方，如积求之。得二百为益实，一百为从方，二为从廉，一为益隅，立方开之，[3] 得黄方。合问。

【注释】

[1] 积指勾股积。此即：$ab-[(a+b)-c]=10$。（郭）

[2] 此即：$a+b=7$。（郭）

[3] 开方式的现代形式为：$-x^3+2x^2+100x-200=0$。（陈）

【今译】

今有勾股之积减去弦和较，余10步。只云勾股和是7步。问：黄方为多少？

答：2步。

术：设天元一为黄方，以如积方法求其解。得到−200为常数项，100为一次项系数，2为二次项系数，−1为最高次项系数，开立方，便得到黄方。符合所问。

12 for the negative last *lian*, and 1 for the positive *yu*, an expression[3] of the fourth degree whose root is the required *xian*.

[Notes]

[1] That is, $[(a+b)-c][(a+b)+c]+(c-a)(c+a)=40$. (G)

[2] That is, $a^2=c+b$. (G)

[3] The expression in modern form is the equation: $x^4-12x^3-54x^2+140x+1525=0$. (C)

4. The *ji* minus the *xian he jiao* is equal to 10 *bu* [1], and the sum of the *gou* and the *gu* is 7 *bu* [2]. Find the *huang fang*.

Ans. 2 *bu*.

Process. Let the element *tian* be the *huang fang*. From the statement we have 200 for the negative *shi*, 100 for the positive *fang*, 2 for the positive *lian*, and 1 for the negative *yu*, a cubic expression,[3] whose root is the required *huang fang*.

[Notes]

[1] The *ji* means the product of the *gou* and the *gu*. That is, $ab-[(a+b)-c]=10$. (G)

[2] That is, $a+b=7$. (G)

[3] The expression in modern form is the equation: $-x^3+2x^2+100x-200=0$. (C)

5.

【原文】

今有积加平幂，减长幂，以平幂乘之，减和幂，余不及积幂八千四百六十步。[1] 只云长平较三步。[2] 问：长、平各几何？

答曰：平九步，长一十二步。

术曰：立天元一为长，如积求之。得八千五百三十二为正实，一百二十三为益方，五十九为从廉，九为益隅，立方开之，[3] 得长。合问。

【注释】

[1] 设 a 为平，b 为长，积指长、平之积 ab，和指长、平之和 $a+b$。此即：$(ab)^2-[(ab+a^2-b^2)a^2-(a+b)^2]=8460$。（郭）

[2] 此即：$b-a=3$。（郭）

[3] 开方式的现代形式为：$-9x^3+59x^2-123x+8532=0$。（陈）

【今译】

今有长、平之积加平之幂，减长之幂，以平之幂乘之，减去长、平和之幂，其余比长、平之积的幂少8460步。只云长与平之差为3步。问：长、平各为多少？

答：平为9步，长为12步。

术：设天元一为长，以如积方法求其解。得到8532为常数项，−123为一次项系数，59为二次项系数，−9为最高次项系数，开立方，便得到长。符合所问。

6.

【原文】

今有积加长，以半平乘之，得一千九百五十步。[1] 只云长五分之三减平三分之二，余七步。[2] 问：长、平各几何？

答曰：平一十二步，长二十五步。

术曰：立天元一为长，如积求之。得一十三万为益实，三千三百二十

5. If the *ji* plus the square of the width minus the square of the length be multiplied by the square of the width, minus the square of the *he* the result is less than the square of the *ji* by 8460 *bu*.[1] The difference between the length and the width is 3 *bu*[2]. Find the length and the width.

Ans. Width, 9 *bu*; length, 12 *bu*.

Process. Let the element *tian* be the length. From the statement we have 8532 for the positive *shi*, 123 for the negative *fang*, 59 for the positive *lian*, and 9 for the negative *yu*, a cubic expression[3] whose root is the required length.

[Notes]

[1] Let a be the width, b the length. The *ji* means the product of the width and the length, ab. The *he* means the sum of the width and the length, $a + b$. That is, $(ab)^2 - [(ab + a^2 - b^2)a^2 - (a + b)^2] = 8460$. (G)

[2] That is, $b - a = 3$. (G)

[3] The expression in modern form is the equation: $-9x^3 + 59x^2 - 123x + 8532 = 0$. (C)

6. The sum of the *ji* and the length multiplied by one half of the width is equal to 1950 *bu*.[1] From three-fifths of the length subtract two-thirds of the width, the remainder is 7 *bu*.[2] Find the length and the width.

Ans. Width, 12 *bu*; length, 25 *bu*.

Process. Let the element *tian* be the length. From the statement we have

五为从方，六百为益廉，二十七为从隅，立方开之，[3] 得长。合问。

【注释】

[1] 此即：$(ab+b)\times\frac{1}{2}a=1950$。（郭）

[2] 此即：$\frac{3}{5}b-\frac{2}{3}a=7$。（郭）

[3] 开方式的现代形式为：$27x^3-600x^2+3325x-130000=0$。（陈）

【今译】

今有长、平之积加长，以平的一半乘之，得到1950步。只云长的$\frac{3}{5}$减去平的$\frac{2}{3}$，余7步。问：长、平各为多少？

答：平为12步，长为25步。

术：设天元一为长，以如积方法求其解。得到-130000为常数项，3325为一次项系数，-600为二次项系数，27为最高次项系数，开立方，便得到长。符合所问。

7.

【原文】

今有积加三较，以长乘之，减三平，余九千七百四十四步。[1] 只云长取太半，平取弱半，为共，不及一长四步。[2] 问：长、平各几何？

答曰：平一十六步，长二十四步。

术曰：立天元一为长，如积求之。得二万九千八十八为益实，一百三十二为从方，五十一为益廉，四为正隅，立方开之，[3] 得长。合问。

【注释】

[1] 较指长、平之差$b-a$。此即：$[ab+3(b-a)]b-3a=9744$。（郭）

[2] 太半即$\frac{2}{3}$，弱半即$\frac{1}{4}$。此即：$b-(\frac{2}{3}b+\frac{1}{4}a)=4$。（郭）

[3] 开方式的现代形式为：$4x^3-51x^2+132x-29088=0$。（陈）

【今译】

今有长、平之积加三倍的长、平之差，以长乘之，减去3倍的平，余9744步。只云取长的$\frac{2}{3}$，平的$\frac{1}{4}$，二者之和比长少4步。问：长、平各为多少？

答：平为16步，长为24步。

术：设天元一为长，以如积方法求其解。得到-29088为常数项，132为一次项系数，-51为二次项系数，4为最高次项系数，开立方，便得到长。符合所问。

130000 for the negative *shi*, 3325 for the positive *fang*, 600 for the negative *lian*, and 27 for the positive *yu*, a cubic expression [3] whose root is the required length.

[Notes]

[1] That is, $(ab + b) \times \frac{1}{2}a = 1950$. (G)

[2] That is, $\frac{3}{5}b - \frac{2}{3}a = 7$. (G)

[3] The expression in modern form is the equation: $27x^3 - 600x^2 + 3325x - 130000 = 0$. (C)

7. Multiply the *ji* plus the three *jiao* by the length and from the result subtract three times the width. The remainder is equal to 9744 *bu*.[1] The sum of the great half of the length and the weak half of the width is less than the length by 4 *bu*.[2] Find the length and the width.

Ans. Width, 16 *bu*; length, 24 *bu*.

Process. Let the element *tian* be the length. From the statement we have 29088 for the negative *shi*, 132 for the positive *fang*, 51 for the negative *lian*, and 4 for the positive *yu*, a cubic expression [3] whose root is the required length.

[Notes]

[1] The *jiao* means the difference between the length and the width, $b - a$. That is, $[ab + 3(b - a)]b - 3a = 9744$. (G)

[2] A great half is $\frac{2}{3}$ and a weak half is $\frac{1}{4}$. That is, $b - (\frac{2}{3}b + \frac{1}{4}a) = 4$. (G)

[3] The expression in modern form is the equation: $4x^3 - 51x^2 + 132x - 29088 = 0$. (C)

8.

【原文】

今有积幂减平，余一万一千六百五十五步。[1] 只云长四分之一、平三分之一、和二分之一，共得一十六步二分步之一。[2] 问：长、平各几何？

答曰：平九步，长一十二步。

术曰：立天元一为平，如积求之。得九十四万四千五十五为益实，八十一为益方，三万九千二百四为从上廉，三千九百六十为益下廉，一百为正隅，三乘方开之，[3] 得平。合问。

【注释】

[1] 此即：$(ab)^2 - a = 11655$。（郭）

[2] 此即：$\frac{1}{4}b + \frac{1}{3}a + \frac{1}{2}(a + b) = 16\frac{1}{2}$。（郭）

[3] 开方式的现代形式为：$100x^4 - 3960x^3 + 39204x^2 - 81x - 944055 = 0$。（陈）

【今译】

今有平、长之积的幂减去平，余 11655 步。只云长的$\frac{1}{4}$，平的$\frac{1}{3}$，平、长之和的$\frac{1}{2}$，共为 $16\frac{1}{2}$ 步。问：长、平各为多少？

答：平为 9 步，长为 12 步。

术：设天元一为平，以如积方法求其解。得到 −944055 为常数项，−81 为一次项系数，39204 为二次项系数，−3960 为三次项系数，100 为最高次项系数，开四次方，便得到平。符合所问。

9.

【原文】

今有积减较幂，余七十一步。[1] 只云三相和四十步[2]。问：长、平和得几何？

答曰：二十三步。

术曰：立天元一为长、平和，如积求之。得四千七十一为益实，二百为从方，一为益隅，平方开之，[3] 得和。合问。

8. The difference between the square of the *ji* and the width is equal to 11655 *bu*[1], and the sum of one fourth of the length, one third of the width, and one half of the *he* is equal to $16\frac{1}{2}$ *bu*.[2] Find the length and the width.

Ans. Width, 9 *bu*; length, 12 *bu*.

Process. Let the element *tian* be the width. From the statement we have 944055 for the negative *shi*, 81 for the negative *fang*, 39204 for the positive first *lian*, 3960 for the negative last *lian*, and 100 for the positive *yu*. Solving this expression[3] of the fourth degree we have the required width.

[Notes]

[1] That is, $(ab)^2 - a = 11655$. (G)

[2] That is, $\frac{1}{4}b + \frac{1}{3}a + \frac{1}{2}(a+b) = 16\frac{1}{2}$. (G)

[3] The expression in modern form is the equation: $100x^4 - 3960x^3 + 39204x^2 - 81x - 944055 = 0$. (C)

9. The difference between the *ji* and the square of the *jiao* is 71 *bu*[1] and the sum of the three sides is 40 *bu*[2]. Find the sum of the length and the width.

Ans. 23 *bu*.

Process. Let the element *tian* be the sum of the length and the width. From the statement we have 4071 for the negative *shi*, 200 for the positive *fang*, and 1 for the negative *yu*, a quadratic expression[3] whose root is the required sum.

【注释】

[1] 此即：$ab-(b-a)^2=71$。（郭）

[2] 三相和是以平为勾，以长为股的勾股形的勾、股、弦三者之和。此即：$a+b+\sqrt{a^2+b^2}=40$。（郭）

[3] 开方式的现代形式为：$-x^2+200x-4071=0$。（陈）

【今译】

今有长、平之积减去长、平之差的幂，余71步。只云以平为勾，以长为股的勾股形的三边之和为40步。问：长、平之和为多少？

答：23步。

术：设天元一为长、平之和，以如积方法求其解。得到-4071为常数项，200为一次项系数，-1为最高次项系数，开平方，便得到长、平之和。符合所问。

10.

【原文】

今有积加和，以积乘之，得二千一百二十步。[1] 只云长多于平三步[2]。问：积几何？

答曰：四十步。

术曰：立天元一为直积，如积求之。得四百四十九万四千四百为正实，四千二百四十九为益上廉，四为益下廉，一为正隅，三乘方开之，[3] 得积。合问。

【注释】

[1] 此即：$[ab+(a+b)]ab=2120$。（郭）

[2] 此即：$b-a=3$。（郭）

[3] 开方式的现代形式为：$x^4-4x^3-4249x^2+4494400=0$。（陈）

【今译】

今有长、平之积加长、平之和，以长、平之积乘之，得2120步。只云长多于平3步。问：长、平之积为多少？

答：40步。

术：设天元一为长、平之直积，以如积方法求其解。得到4494400为常数项，-4249为二次项系数，-4为三次项系数，1为最高次项系数，开四次方，便得到长、平之积。符合所问。

[Notes]

[1] That is, $ab - (b - a)^2 = 71$. (G)

[2] The sum of the three sides is the sum of the *gou*, the *gu*, and the *xian* of a right triangle. The width is used for the *gou*, and the length for the *gu*. That is, $a + b + \sqrt{a^2 + b^2} = 40$. (G)

[3] The expression in modern form is the equation: $-x^2 + 200x - 4071 = 0$. (C)

10. The product of the sum of the *ji* and *he* by the *ji* is equal to 2120 *bu*[1], and the length exceeds the width by 3 *bu*[2]. Find the *ji*.

Ans. 40 *bu*.

Process. Let the element *tian* be the *zhi ji*. From the statement we have 4494400 for the positive *shi*, 4249 for the negative first *lian*, 4 for the negative last *lian*, and 1 for the positive *yu*. Solving this expression[3] of the fourth degree we have the required *ji*.

[Notes]

[1] That is, $[ab + (a + b)]\, ab = 2120$. (G)

[2] That is, $b - a = 3$. (G)

[3] The expression in modern form is the equation:

$x^4 - 4x^3 - 4249x^2 + 4494400 = 0$. (C)

11.

【原文】

今有积加斜幂，得三百三十三步。[1] 只云并长、平、斜得三十六步[2]。问：弦几何？

答曰：一十五步。

术曰：立天元一为弦，如积求之。得三百一十五为正实，三十六为益方，一为正隅，平方开之，[3] 得弦。合问。

【注释】

[1] 斜幂即弦幂。此即：$ab + c^2 = 333$。(郭)

[2] 此即：$a + b + c = 36$。(郭)

[3] 开方式的现代形式为：$x^2 - 36x + 315 = 0$。(陈)

【今译】

今有长、平之积加斜幂，得333步。只云长、平、斜三者之和为36步。问：弦为多少？

答：15步。

术：设天元一为弦，以如积方法求其解。得到315为常数项，-36为一次项系数，1为最高次项系数，开平方，便得到弦。符合所问。

12.

【原文】

今有积减平，以积乘之，又减五平四积，余二十七万九千六百三十步。[1] 只云长取五分之一，平取三分之二，其长分子数如平分子数二分之一。[2] 问：长、平各几何？

答曰：平一十八步，长三十步。

术曰：立天元一为平，如积求之。得五十万三千三百三十四为益实，九为益方，一十二为益上廉，三为益下廉，五为正隅，三乘方开之，[3] 得平。合问。

11. The sum of the *ji* and the square of the *xie* is equal to 333 *bu* [1], and the sum of the length, the width, and the *xie* is equal to 36 *bu* [2]. Find the *xian*.

Ans. 15 *bu*.

Process. Let the element *tian* be the *xian*. From the statement we have 315 for the positive *shi*, 36 for the negative *fang*, and 1 for the positive *yu*. Solving this quadratic expression [3] we have the required *xian* .

【 Notes 】

[1] The square of the *xie* (the diagonal of rectangle) here means the square of the *xian*. That is, $ab + c^2 = 333$. (G)

[2] That is, $a + b + c = 36$. (G)

[3] The expression in modern form is the equation: $x^2 - 36x + 315 = 0$. (C)

12. Multiply the difference between the *ji* and the width by the *ji* and subtract from the result the sum of 5 times the width and 4 times the *ji*. The remainder is equal to 279630 *bu*.[1] It is said that the value of the numerator of the fraction which is one-fifth of the length, is the same as one-half the numerator of the fraction which is equal to two-thirds of the width.[2] Find the length and the width.

Ans. Width, 18 *bu*; length, 30 *bu*.

Process. Let the element *tian* be the width. From the statement we have

【注释】

[1] 此即：$(ab-a)ab-(5a+4ab)=279630$。（郭）

[2] 长的 $\frac{1}{5}$ 为 $\frac{1}{5}b$，平的 $\frac{2}{3}$ 为 $\frac{2}{3}a$，通分，分别为 $\frac{3b}{15}$，$\frac{10a}{15}$。此即：$3b=\frac{1}{2}(10a)$。（郭）

[3] 开方式的现代形式为：$5x^4-3x^3-12x^2-9x-503334=0$。（陈）

【今译】

今有长、平之积减平，再以积乘之，又减去平的5倍及长、平之积的4倍，余279630步。只云取长的$\frac{1}{5}$，取平的$\frac{2}{3}$，其长的分子数是平的分子数的$\frac{1}{2}$。问：长、平各为多少？

答：平为18步，长为30步。

术：设天元一为平，以如积方法求其解。得到−503334为常数项，−9为一次项系数，−12为二次项系数，−3为三次项系数，5为最高次项系数，开四次方，便得到平。符合所问。

13.

【原文】

今有积幂减二长、一平，余四万六千五百七十八步。[1] 只云平自乘与长等[2]。问：长、平各几何？

答曰：平六步，长三十六步。

术曰：立天元一为平，如积求之。得四万六千五百七十八为益实，一为益方，二为益上廉，一为正隅，五乘方开之，[3] 得平。合问。

【注释】

[1] 此即：$(ab)^2-(2b+a)=46578$。（郭）

[2] 此即：$a^2=b$。（郭）

[3] 开方式的现代形式为：$x^6-2x^2-x-46578=0$。（陈）

【今译】

今有长、平之积的幂减去长的2倍及平，余46578步。只云平自乘与长相等。问：长、平各为多少？

答：平为6步，长为36步。

术：设天元一为平，以如积方法求其解。得到−46578为常数项，−1为一次项系数，−2为二次项系数，1为最高次项系数，开六次方，便得到平。符合所问。

503334 for the negative *shi*, 9 for the negative *fang*, and 12 for the negative first *lian*, 3 for the negative last *lian*, and 5 for the positive *yu*. Solving this biquadratic expression[3] we have the required width.

[Notes]

[1] That is, $(ab - a)ab - (5a + 4ab) = 279630$. (G)

[2] One-fifth of the length is $\frac{1}{5}b$, and two-thirds of the width is $\frac{2}{3}a$. Make their denominators equal, then they become $\frac{3b}{15}$ and $\frac{10a}{15}$. That is, $3b = \frac{1}{2}(10a)$. (G)

[3] The expression in modern form is the equation: $5x^4 - 3x^3 - 12x^2 - 9x - 503334 = 0$. (C)

13. The sum of twice the length and the width subtracted from the square of the *ji* is equal to 46578 *bu*[1]; the square of the width is equal to the length[2]. Find the length and the width.

Ans. Width, 6 *bu*; length, 36 *bu*.

Process. Let the element *tian* be the width. From the statement we have 46578 for the negative *shi*, 1 for the negative *fang*, 2 for the negative first *lian*, and 1 for the positive *yu*. Solving this expression[3] of the sixth degree we have the required width.

[Notes]

[1] That is, $(ab)^2 - (2b + a) = 46578$. (G)

[2] That is, $a^2 = b$. (G)

[3] The expression in modern form is the equation: $x^6 - 2x^2 - x - 46578 = 0$. (C)

14.

【原文】

今有积加一长、二平、三和、四较，自乘，减一和、二较、三平、四长，余一十五万五千八百五步。[1] 只云平幂与较等[2]。问：长、平各几何？

答曰：平五步，长三十步。

术曰：立天元一为平，如积求之。得一十五万五千八百五为益实，九为益方，七十四为从上廉，一百六十二为从二廉，九十九为从三廉，一十八为从下廉，一为正隅，五乘方开之，[3] 得平。合问。

【注释】

[1] 此即：$[ab + b + 2a + 3(a + b) + 4(b - a)]^2 - [(a + b) + 2(b - a) + 3a + 4b] = 155805$。（郭）

[2] 此即：$a^2 = b - a$。（郭）

[3] 开方式的现代形式为：$x^6 + 18x^5 + 99x^4 + 162x^3 + 74x^2 - 9x - 155805 = 0$。（陈）

【今译】

今有长、平之积加1倍的长，2倍的平，3倍的长、平之和，4倍的长、平之差，其自乘，减去1倍的长、平之和，2倍的长、平之差，3倍的平，4倍的长，余155805步。只云平幂与长、平之差相等。问：长、平各为多少？

答：平为5步，长为30步。

术：设天元一为平，以如积方法求其解。得到−155805为常数项，−9为一次项系数，74为二次项系数，162为三次项系数，99为四次项系数，18为五次项系数，1为最高次项系数，开六次方，便得到平。符合所问。

14. From the square of the sum of the *ji*, the length, twice the width, three times the *he*, and four times the *jiao* subtract the sum of the *he*, twice the *jiao*, three times the width, and four times the length. The remainder is equal to 155805 *bu*.[1] The square of the width is equal to the *jiao*[2]. Find the length and the width.

Ans. Width, 5 *bu*; length, 30 *bu*.

Process. Let the element *tian* be the width. From the statement we have 155805 for the negative *shi*, 9 for the negative *fang*, 74 for the positive third *lian*, 162 for the positive second *lian*, 99 for the positive third *lian*, 18 for the positive last *lian*, and 1 for the positive *yu*. Solving the root of this expression[3] of the sixth degree we have the required width.

【 Notes 】

[1] That is, $[ab + b + 2a + 3(a + b) + 4(b - a)]^2 - [(a + b) + 2(b - a) + 3a + 4b] = 155805$. (G)

[2] That is, $a^2 = b - a$. (G)

[3] The expression in modern form is the equation: $x^6 + 18x^5 + 99x^4 + 162x^3 + 74x^2 - 9x - 155805 = 0$. (C)

15.

【原文】

今有积加三平，减一较，余自乘，减三平，加一较，得七万八千四百一十四步。[1] 只云平自乘与和等[2]。问：长、平各几何？

答曰：平七步，长四十二步。

术曰：立天元一为平，如积求之。得七万八千四百一十四为益实，五为益方，二十六为从上廉，二十为益二廉，一十四为从三廉，四为益下廉，一为正隅，五乘方开之，[3] 得平。合问。

【注释】

[1] 此即：$[ab+3a-(b-a)]^2-3a+(b-a)=78414$。（郭）

[2] 此即：$a^2=a+b$。（郭）

[3] 开方式的现代形式为：$x^6-4x^5+14x^4-20x^3+26x^2-5x-78414=0$。（陈）

【今译】

今有长、平之积加3倍的平，减1倍的长、平之差，余数自乘，减去3倍的平，加1倍的长、平之差，得78414步。只云平自乘与长、平之和相等。问：长、平各为多少？

答：平为7步，长为42步。

术：设天元一为平，以如积方法求其解。得到−78414为常数项，−5为一次项系数，26为二次项系数，−20为三次项系数，14为四次项系数，−4为五次项系数，1为最高次项系数，开六次方，便得到平。符合所问。

16.

【原文】

今有积加平，以长中半乘之，得三千九百步。[1] 只云长以平方开之，所得不及平七步。[2] 问：长、平各几何？

答曰：平一十二步，长二十五步。

术曰：立天元一为平，如积求之。得七千八百为益实，二千四百五十为从方，一千三百八十六为益上廉，二百九十五为从二廉，二十八为益下廉，一为正隅，四乘方开之，[3] 得平。合问。

15. Add to the *ji* three times the width and subtract from the sum the *jiao*; after squaring the remainder subtract the sum of three times the width and the *jiao*. The result is equal to 78414 *bu*.[1] The square of the width is equal to the *he*[2]. Find the length and the width.

Ans. Width, 7 *bu*; length, 42 *bu*.

Process. Let the element *tian* be the width. From the statement we have 78414 for the negative *shi*, 5 for the negative *fang*, 26 for the positive first *lian*, 20 for the negative second *lian*, 14 for the positive third *lian*, 4 for the negative last *lian*, and 1for the positive *yu*, an expression[3] of the sixth degree whose root is the required width.

[Notes]

[1] That is, $[ab + 3a - (b - a)]^2 - 3a + (b - a) = 78414$. (G)

[2] That is, $a^2 = a + b$. (G)

[3] The expression in modern form is the equation: $x^6 - 4x^5 + 14x^4 - 20x^3 + 26x^2 - 5x - 78414 = 0$. (C)

16. The product of the sum of the *ji* and the width by one-half of the length is equal to 3900 *bu*.[1] It is said that the square root of the length is less than the width by 7 *bu*.[2] Find the length and the width.

Ans. Width, 12 *bu*; length, 25 *bu*.

Process. Let the element *tian* be the width. From the statement we have 7800 for the negative *shi*, 2450 for the positive *fang*, 1386 for the negative first *lian*, 295 for the positive second *lian*, 28 for the negative last *lian*, and

【注释】

[1] 中半即半。此即：$(ab+a)\times\frac{1}{2}b=3900$。（郭）

[2] 此即：$a-\sqrt{b}=7$。（郭）

[3] 开方式的现代形式为：$x^5-28x^4+295x^3-1386x^2+2450x-7800=0$。（陈）

【今译】

今有长、平之积加平，以长的一半乘之，得3900步。只云长的平方根比平少7步。问：长、平各为多少？

答：平为12步，长为25步。

术：设天元一为平，以如积方法求其解。得到-7800为常数项，2450为一次项系数，-1386为二次项系数，295为三次项系数，-28为四次项系数，1为最高次项系数，开五次方，便得到平。符合所问。

17.

【原文】

今有积加一和、三较，以积乘之，减一长、二较，又长乘之，得一十四万七千二百一十六步。[1] 只云平以立方开之，如长六分之一。[2] 问：长、平各几何？

答曰：平八步，长一十二步。

术曰：立天元一为开方数，如积求之。得一万二千二百六十八为益实，九为益上廉，一为从三廉，七十二为从五廉，六为益七廉，一十八为正隅，八乘方开之，[3] 得二步，即开方数。六之，得长。合问。

【注释】

[1] 此即：$\{[ab+(a+b)+3(b-a)](ab)-[b+2(b-a)]\}b=147216$。（郭）

[2] 此即：$\sqrt[3]{a}=\frac{1}{6}b$。（郭）

[3] 开方式的现代形式为：$18x^9-6x^8+72x^6+x^4-9x^2-12268=0$。（陈）

【今译】

今有长、平之积加1倍的长、平之和，与3倍的长、平之差，以积乘之，减1倍的长与2倍的长、平之差，又以长乘之，得147216步。只云平的立方根等于长的$\frac{1}{6}$。问：长、平各为多少？

答：平为8步，长为12步。

术：设天元一为开方数，以如积方法求其解。得到-12268为常数项，-9为二次项系数，1为四次项系数，72为六次项系数，-6为八次项系数，18为最高次项系数，开九次方，得到2步，就是开方数。以6乘，便得到长。符合所问。

1 for the positive *yu*. Solving this expression[3] of the fifth degree we have the required width.

[Notes]

[1] The middle half is one-half. That is $(ab + a) \times \frac{1}{2}b = 3900$. (G)

[2] That is, $a - \sqrt{b} = 7$. (G)

[3] The expression in modern form is the equation: $x^5 - 28x^4 + 295x^3 - 1386x^2 + 2450x - 7800 = 0$. (C)

17. Multiply the sum of the *ji*, the *he* and three times the *jiao* by the *ji* and subtract the length and twice the *jiao*; multiply this remainder by the length the product is equal to 147216 *bu*[1]. It is said that the cube root of the width is equal to one-sixth the length[2]. Find the length and the width.

Ans. Width, 8 *bu*; length, 12 *bu*.

Process. Let the element *tian* be the cube root of the width. From the statement we have 12268 for the negative *shi*, 9 for the negative first *lian*, 1 for the positive third *lian*, 72 for the positive fifth *lian*, 6 for the negative seventh *lian*, and 18 for the positive *yu*. Solving this expression[3] of the ninth degree we have 2 for its root, 6 times this root is the required length.

[Notes]

[1] That is, $\{[ab + (a + b) + 3(b - a)](ab) - [b + 2(b - a)]\}b = 147216$. (G)

[2] That is, $\sqrt[3]{a} = \frac{1}{6}b$. (G)

[3] The expression in modern form is the equation: $18x^9 - 6x^8 + 72x^6 + x^4 - 9x^2 - 12268 = 0$. (C)

18.

【原文】

今有积，以和乘之，减积，余，以平乘之，加和，得一十七万七千一百六十二步。[1] 只云和为益实，四为益方，三为从上廉，二为益下廉，一为正隅，三乘方开之，如平四分之一。[2] 问：长、平各几何？

答曰：平一十二步，长三十步。

术曰：立天元一为开方数，如积求之。得一十七万七千一百六十二为益实，四为益方，三为从上廉，一百二十六为从二廉，四百六十五为从三廉，五百四十四为益四廉，五百一十二为从五廉，三百八十四为益六廉，一百六十为从七廉，六十四为益下廉，一十六为正隅，九乘方开之，[3] 得三步，为开方数。四之，即平。合问。

【注释】

[1] 此即：$[ab(a+b)-ab]a+(a+b)=177162$。（郭）

[2] 此即：$w^4-2w^3+3w^2-4w-(a+b)=0$ 的根 $w=\frac{1}{4}a$。（郭）

[3] 开方式的现代形式为：

$16x^{10}-64x^9+160x^8-384x^7+512x^6-544x^5+465x^4+126x^3+3x^2-4x-177162=0$。（陈）

【今译】

今有长、平之积，以长、平之和乘之，减去长、平之积，又以平乘之，加长、平之和，得 177162 步。只云以长、平之和为负常数项，−4 为一次项系数，3 为二次项系数，−2 为三次项系数，1 为最高次项系数，开四次方，其根等于平的 $\frac{1}{4}$。问：长、平各为多少？

答：平为 12 步，长为 30 步。

术：设天元一为开方数，以如积方法求其解。得到 −177162 为常数项，−4 为一次项系数，3 为二次项系数，126 为三次项系数，465 为四次项系数，−544 为五次项系数，512 为六次项系数，−384 为七次项系数，160 为八次项系数，−64 为九次项系数，16 为最高次项系数，开十次方，得到 3 步，就是开方数。以 4 乘，便得到平。符合所问。

18. If the product of the *ji* and the *he* minus the *ji* be multiplied by the width and the *he* be added the result is equal to 177162 *bu*.[1] It is said if we take the *he* for the negative *shi*, 4 for the negative *fang*, 3 for the positive first *lian*, 2 for the negative last *lian*, and 1 for the positive *yu* of an expression of the fourth degree the root of the expression is equal to one-fourth of the width.[2] Find the length and the width.

Ans. Width, 12 *bu*; length, 30 *bu*.

Process. Let the element *tian* be the root of the given expression. From the statement we have 177162 for the negative *shi*, 4 for the negative *fang*, 3 for the positive first *lian*, 126 for the positive second *lian*, 465 for the positive third *lian*, 544 for the negative fourth *lian*, 512 for the positive fifth *lian*, 384 for the negative sixth *lian*, 160 for the positive seventh *lian*, 64 for the negative last *lian*, and 16 for the positive *yu*, an expression[3] of the tenth degree with 3 for its root. Multiplying this root by 4 we have the required width.

【Notes】

[1] That is, $[ab(a+b)-ab]a+(a+b)=177162$. (G)

[2] That is, $w=\frac{1}{4}a$ is one root of the equation $w^4-2w^3+3w^2-4w-(a+b)=0$. (G)

[3] The expression in modern form is the equation: $16x^{10}-64x^9+160x^8-384x^7+512x^6-544x^5+465x^4+126x^3+3x^2-4x-177162=0$. (C)

混积问元 一十八问

1.

【原文】

今有直田积，加斜幂，减平幂，余半之，复减斜幂，余六十七步。[1] 只云斜较相和二十步[2]。问：斜长几何？

答曰：一十三步。

术曰：立天元一为斜长，如积求之。得二万二千一百七十八为正实，五千三百二十为益方，四百九十九为从上廉，三十为益下廉，一为正隅，三乘方开之，[3] 得斜。合问。

【注释】

[1] 设直田宽即平为 a，长为 b，斜为 c。此即：$c^2-\frac{1}{2}(ab+c^2-a^2)=67$。（郭）

[2] 斜较相和即斜与长、平之差的和。此即：$c+(b-a)=20$。（郭）

[3] 开方式的现代形式为：$x^4-30x^3+499x^2-5320x+22178=0$。（陈）

【今译】

今有直田的面积加斜幂，减去平幂，取其余数的一半，再去减斜幂，余 67 步。只云直田的斜与长、平之差的和是 20 步。问：直田的斜长为多少？

答：13 步。

术：设天元一为直田的斜长，以如积方法求其解。得到 22178 为常数项，−5320 为一次项系数，499 为二次项系数，−30 为三次项系数，1 为最高次项系数，开四次方，便得到斜长。符合所问。

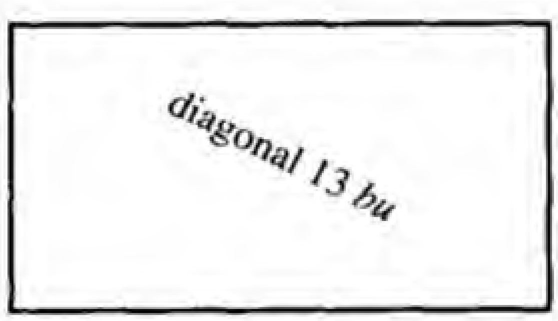
Hun Ji Wen Yuan (Problems on Plane Figures)

18 Problems

1. Divide the area of a rectangle plus the square of a diagonal minus the square of the width by 2 and subtract from the quotient the square of the diagonal. The result is equal to 67 *bu*.[1] The sum of the diagonal and the *jiao* is equal to 20 *bu*[2]. Find the diagonal.

Ans. 13 *bu*.

Process. Let the element *tian* be a diagonal of the rectangle. From the statement we have 22178 for the positive *shi*, 5320 for the negative *fang*, 499 for the positive first *lian*, 30 for the negative last *lian*, and 1 for the positive *yu*, an expression[3] of the fourth degree whose root is the required diagonal.



[Notes]

[1] Let the width of the rectangle be a, the length b, the diagonal c. That is, $c^2 - \frac{1}{2}(ab + c^2 - a^2) = 67$. (G)

[2] The sum of the diagonal and the *jiao* is the sum of the diagonal and the difference between the length and the width. That is, $c + (b - a) = 20$. (G)

[3] The expression in modern form is the equation: $x^4 - 30x^3 + 499x^2 - 5320x + 22178 = 0$. (C)

2.

【原文】

今有方田幂，加斜长[1]，减方周，余以方面乘之，减方面，余二千七百七十五步。[2] 问：方面几何？

答曰：一十五步。

术曰：立天元一为方面，如积求之。得一万三千八百七十五为益实，五为益方，一十三为益廉，五为正隅，立方开之，[3] 得方面。合问。

方面一十五步

【注释】

[1] 在中国古代方田边与斜的比例为5∶7。（陈）

[2] 设方田边长为a，此即 $(a^2 + \frac{7}{5}a - 4a)a - a = 2775$。（郭）

[3] 开方式的现代形式为：$5x^3 - 13x^2 - 5x - 13875 = 0$。（陈）

【今译】

今有方田的幂加其斜长，减去方田的周长，其余数以边长乘之，减去边长，余2775步。问：方田的边长为多少？

答：15步。

术：设天元一为方田的边长，以如积方法求其解。得到−13875为常数项，−5为一次项系数，−13为二次项系数，5为最高次项系数，开立方，便得到方田的边长。符合所问。

3.

【原文】

今有勾股田积，加弦和和，得一百四十步。[1] 只云勾股较一十七步[2]。问：股几何？

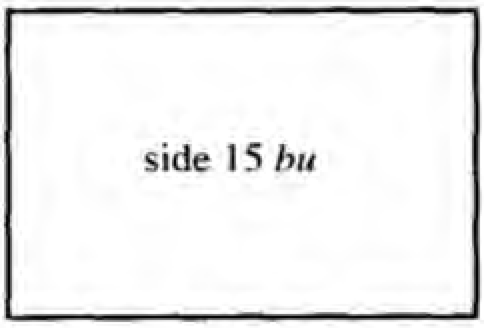
答曰：二十四步。

术曰：立天元一为股，如积求之。得九万七千四百四十为正实，八千三百为从方，四百六十七为益上廉，二十六为益下廉，一为正隅，三乘方开之，[3] 得股。合问。

2. If the square of a side of a square plus a diagonal[1] minus the perimeter be multiplied by the side and then the side subtracted from the product. The result is equal to 2775 *bu*.[2] Find the side of the square.

Ans. 15 *bu*.

Process. Let the element *tian* be a side of a square. From the statement we have 13875 for the negative *shi*, 5 for the negative *fang*, 13 for the negative *lian*, and 5 for the positive *yu*, a cubic expression[3] whose root is the required side of the square.



[Notes]

[1] In ancient times the ratio of a side of a square to a diagonal was taken as 5 to 7. (C)

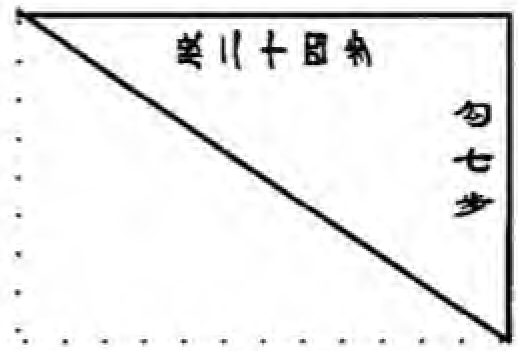
[2] Let a side of the square be a. That is, $(a^2 + \frac{7}{5}a - 4a)a - a = 2775$. (G)

[3] The expression in modern form is the equation: $5x^3 - 13x^2 - 5x - 13875 = 0$. (C)

3. In a right triangle, the sum of the *ji* and the *xian he he* is equal to 140 *bu*;[1] the difference between the *gou* and the *gu* is 17 *bu*[2]. Find the *gu*.

Ans. 24 *bu*.

Process. Let the element *tian* be the *gu*. From the statement we have 97440 for the positive *shi*, 8300 for the positive *fang*, 467 for the negative first *lian*, 26 for the negative last *lian*, and 1 for the positive *yu*, an expression[3]



【注释】

[1] 此即：$\frac{1}{2}ab + [c + (a + b)] = 140$。（郭）

[2] 即：$b - a = 17$。（郭）

[3] 开方式的现代形式为：$x^4 - 26x^3 - 467x^2 + 8300x + 97440 = 0$。（陈）

【今译】

今有勾股田的面积加弦和和，得140步。只云勾股较为17步。问：股为多少？

答：24步。

术：设天元一为股，以如积方法求其解。得到97440为常数项，8300为一次项系数，−467为二次项系数，−26为三次项系数，1为最高次项系数，开四次方，便得到股。符合所问。

4.

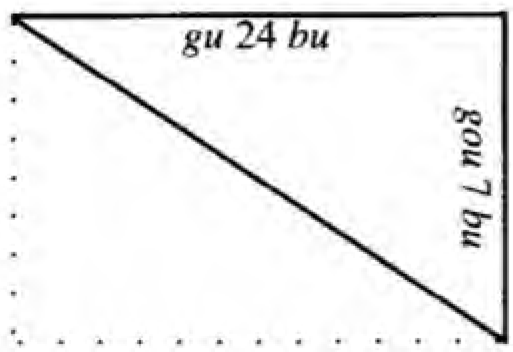
【原文】

今有梯田积，加小阔[1]，减大阔，余以小阔乘之，得四千一百五十二步。[2]只云大阔不及正长九步，却多小阔四步。[3]问：二阔及长各几何？

答曰：大阔一十六步，小阔一十二步，正长二十五步。

术曰：立天元一为大阔，如积求之。得四千六十四为益实，五十为益方，三为从廉，一为正隅，立方开之，[4]得大阔。合问。

of the fourth degree whose root is the required *gu*.



【Notes】

[1] That is, $\frac{1}{2}ab + [c + (a + b)] = 140$. (G)

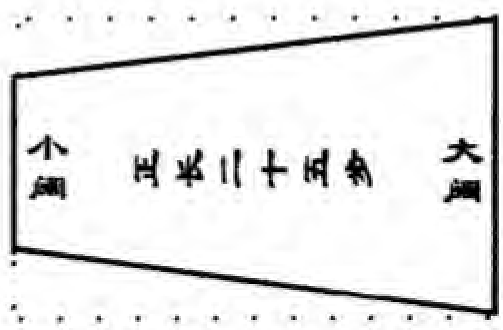
[2] That is, $b - a = 17$. (G)

[3] The expression in modern form is the equation: $x^4 - 26x^3 - 467x^2 + 8300x + 97440 = 0$. (C)

4. The area of a trapezoid plus the *xiao kuo*[1] minus the *da kuo* and the remainder multiplied by the *xiao kuo* is equal to 4152 *bu*.[2] It is said that the *da kuo* is less than the *zheng chang* by 9 *bu*, but exceeds the *xiao kuo* by 4 *bu*.[3] Find the *da kuo*, *xiao kuo*, and *zheng chang*.

Ans. *Da kuo*, 16 *bu*; *xiao kuo*, 12 *bu*; *zheng chang*, 25 *bu*.

Process. Let the element *tian* be the *da kuo* of the trapezoid. From the statement we have 4064 for the negative *shi*, 50 for the negative *fang*, 3 for



【注释】

[1] 小阔为梯形上底，大阔为下底，正长为高。（陈）

[2] 梯田在《九章算术》中称为邪田或箕田。设其面积、小阔、大阔、正长分别为S，a，b，h，《九章算术》给出的面积公式为$S=\frac{1}{2}(a+b)h$。此即：$[\frac{1}{2}(a+b)h+a-b]a=4152$。（郭）

[3] 此即：$h-b=9$，$a+4=b$。（郭）

[4] 开方式的现代形式为：$x^3+3x^2-50x-4064=0$。（陈）

【今译】

今有梯田的面积加小阔，减去大阔，余数，再以小阔乘之，得到4152步。只云大阔比正长少9，比小阔多4步。问：大阔、小阔及正长各为多少？

答：大阔16步，小阔12步，正长25步。

术：设天元一为大阔，以如积方法求其解。得到-4064为常数项，-50为一次项系数，3为二次项系数，1为最高次项系数，开三次方，便得到大阔。符合所问。

5.

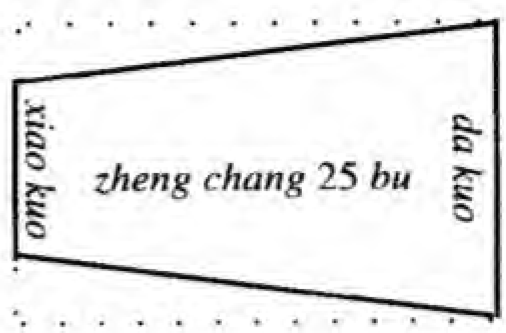
【原文】

今有圭田积，减四长[1]，余五十步。[2] 只云较自乘减阔幂，余与长等。[3] 问：长、阔各几何？

答曰：阔一十二步，长二十五步。

术曰：立天元一为圭长，如积求之。得二百为益实，一十七为益方，一为正隅，平方开之，[4] 得长。合问。

the positive *lian*, and 1 for the positive *yu*, a cubic expression [4] whose root is the required *da kuo*.



[Notes]

[1] The *xiao kuo*, small width, is the upper base; the *da kuo*, great width, the lower base; and the *zheng chang*, the altitude of the trapezoid. (C)

[2] A trapezoid is called *xie tian* or *ji tian* in *The Nine Chapters of Mathematical Procedures*. Let its area, *xiao kuo*, *da kuo*, and *zheng chang* be S, a, b, h, the area formula in *The Nine Chapters of Mathematical Procedures* is $S = \frac{1}{2}(a + b)h$. That is, $[\frac{1}{2}(a + b)h + a - b]a = 4152$. (G)

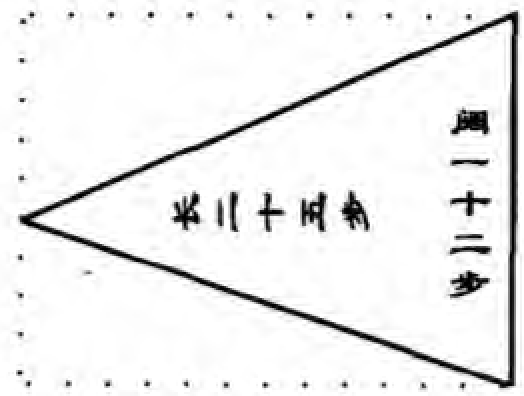
[3] That is, $h - b = 9$, $a + 4 = b$. (G)

[4] The expression in modern form is the equation: $x^3 + 3x^2 - 50x - 4064 = 0$. (C)

5. The difference between the area of a *gui tian* and four times the length [1] is equal to 50 *bu* [2]; and the difference between the square of the *jiao* and the square of the width is equal to the length [3]. Find the length and the width.

Ans. Width, 12 *bu*; length, 25 *bu*.

Process. Let the element *tian* be the length of the *gui tian*. From the statement we have 200 for the negative *shi*, 17 for the negative *fang*,



【注释】

[1] 圭田的长是等腰三角形的高，阔为底，较是高与底的差。（陈）

[2] 记其面积、阔、长分别为 S，a，b，《九章算术》方田章给出面积公式：$S=\frac{1}{2}ab$。此即：$\frac{1}{2}ab-4b=50$。（郭）

[3] 此即：$(b-a)^2-a^2=b$。（郭）

[4] 开方式的现代形式为：$x^2-17x-200=0$。（陈）

【今译】

今有圭田的面积减去4倍的长，余50步。只云圭田的长、阔之差自乘，减去阔幂，余数与长相等。问：长、阔各为多少？

答：阔12步，长25步。

术：设天元一为圭田的长，以如积方法求其解。得到 -200 为常数项，-17 为一次项系数，1为最高次项系数，开平方，便得到圭田的长。符合所问。

6.

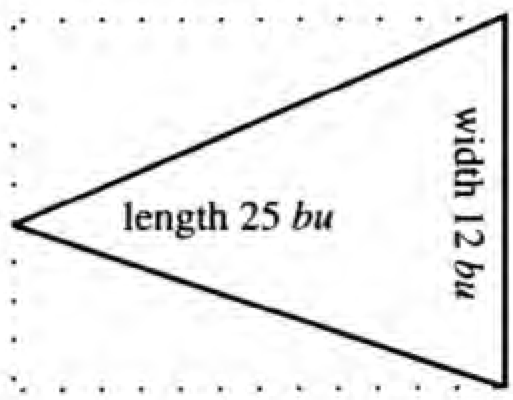
【原文】

今有梭田积，加广幂，减于长幂，不足三十六步。[1] 只云长内虚加一算，平方开之，得数，以减半广，不足四步。[2] 问：长、广各几何？

答曰：广一十八步，长二十四步。

术曰：立天元一为半广，如积求之。得一百八十九为正实，二百五十五为益方，九十八为从上廉，一十七为益下廉，一为正隅，三乘方开之，[3] 得半广。合问。

and 1 for the positive *yu*, a quadratic expression [4] whose root is the required length.



[Notes]

[1] The length of the *gui tian* is the altitude of the isosceles triangle; the width the base , and the *jiao* the difference between the altitude and the base. (C)

[2] Let its area, width, and length be S, a, b. The area formula was given by the chapter of *fang tian*(surveying of land)of *The Nine Chapters of Mathematical Procedures*: $S = \frac{1}{2}ab$. That is, $\frac{1}{2}ab - 4b = 50$. (G)

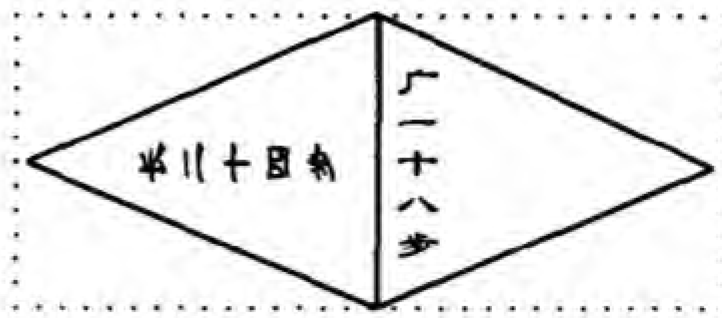
[3] That is, $(b - a)^2 - a^2 = b$. (G)

[4] The expression in modern form is the equation: $x^2 - 17x - 200 = 0$. (C)

6. The sum of the area of a *suo tian* and the square of the width is less than the square of the length by 36 *bu*,[1] and 1 be added to the length, the square root of the sum is less than one-half of the width by 4 *bu* [2]. Find the length and the width.

Ans. Width, 18 *bu*; length, 24 *bu*.

Process. Let the element *tian* be one-half of the width of the *suo tian*. From the statement we have 189 for the positive *shi*, 255 for the negative *fang*,



【注释】

[1] 梭田即菱形。广为短对角线，长为长对角线。设梭田广为a，长为b，显然其面积公式为$S=\frac{1}{2}ab$。此即：$b^2-(\frac{1}{2}ab+a^2)=36$。（郭）

[2] 此即：$\frac{1}{2}a-\sqrt{b+1}=4$。（郭）

[3] 开方式的现代形式为：$x^4-17x^3+98x^2-255x+189=0$。（陈）

【今译】

今有梭田的面积加广幂，减于长幂，不足36步。只云梭田的长加1，其平方根去减广的一半，不足4步。问：梭田的长、广各为多少？

答：广18步，长24步。

术：设天元一为梭田的广的一半，以如积方法求其解。得到189为常数项，−255为一次项系数，98为二次项系数，−17为三次项系数，1为最高次项系数，开四次方，便得到梭田的广的一半。符合所问。

7.

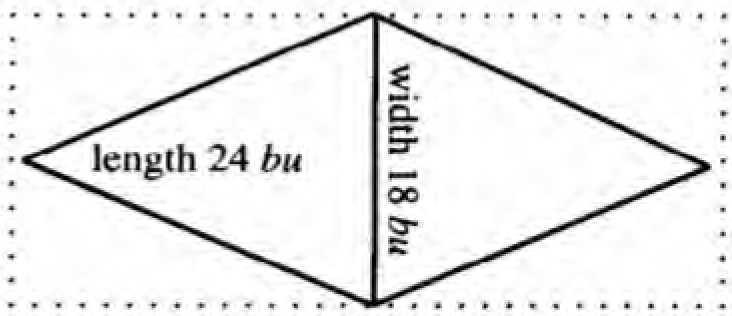
【原文】

今有三斜田积，减中股，余七十六步。[1] 只云中斜多于中股九步，中股不及小斜二步。[2] 问：中股几何？

答曰：八步。

术曰：立天元一为中股，如积求之。得一亿三千三百四十四万八千七百四为益实，七百二万三千六百一十六为益方，八十四万三千二百九十六为从上廉，二十七万八千七百六十八为从二廉，五千三百七十一步七分五厘[3] 为从三廉，四百九十五为益下廉，四十九为益隅，五乘方开之，[4] 得中股。合问。

98 for the positive first *lian*, 17 for the negative last *lian*, and 1 for the positive *yu*, an expression [3] of the fourth degree whose root is the one-half of the required width.



[Notes]

[1] A *suo tian*, a piece of land in the form of a shuttle, is a rhomboid. The width is the short diagonal, and the length the long diagonal. Let the width of the *suo tian* be a, and the length b. Obviously, the area formula is $S = \frac{1}{2}ab$. That is, $b^2 - (\frac{1}{2}ab + a^2) = 36$. (G)

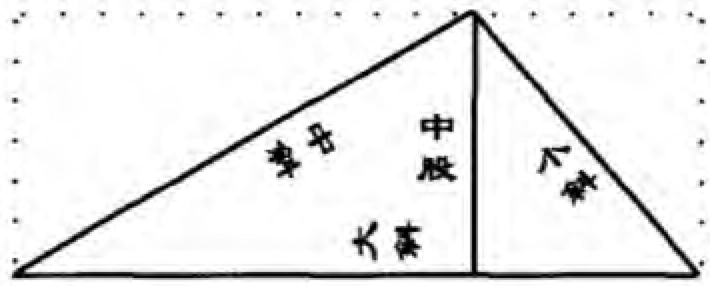
[2] That is, $\frac{1}{2}a - \sqrt{b+1} = 4$. (G)

[3] The expression in modern form is the equation: $x^4 - 17x^3 + 98x^2 - 255x + 189 = 0$. (C)

7. The area of a *san xie tian* minus the *zhong gu* equals 76 *bu*.[1] It is said that the *zhong xie* exceeds the *zhong gu* by 9 *bu*; and the *zhong gu* is less than the *xiao xie* by 2 *bu*.[2] Find the *zhong gu*.

Ans. 8 *bu*.

Process. Let the element *tian* be the *zhong gu*. From the statement we have 133448704 for the negative *shi*, 7023616 for the negative *fang*, 843296 for the positive first *lian*, 278768 for the positive second *lian*, 5371 *bu* 7 *fen* 5 *li* [3] for the positive third *lian*, 495 for the negative last *lian*, and 49 for the



【注释】

［1］三斜田即不等边三角形；中股为高；中斜为三边中小于大边而大于小边的边；大斜为大边；小斜为小边。设三斜田的大斜为 a，中斜为 b，小斜为 c，中股为 h，面积为 S。此即：$S - h = 76$。（郭）

［2］此即：$b - h = 9$，$c - h = 2$。（郭）

［3］五千三百七十一步七分五厘指5371.75步。凡"步"下之分、厘，均指十进小数。以"分"、"厘"、"毫"、"丝"表示整数单位之下的十进小数的方法源于《孙子算经》。（郭）

［4］开方式的现代形式为：$-49x^6 - 495x^5 + 5371.75x^4 + 278768x^3 + 843296x^2 - 7023616x - 133448704 = 0$。（陈）

【今译】

今有三斜田的面积减中股，余76步。只云中斜多于中股9步，中股少于小斜2步。问：中股为多少？

答：8步。

术：设天元一为中股，以如积方法求其解。得到 −133448704 为常数项，−7023616 为一次项系数，843296 为二次项系数，278768 为三次项系数，5371.75 为四次项系数，−495 为五次项系数，−49 为最高次项系数，开六次方，便得到中股。符合所问。

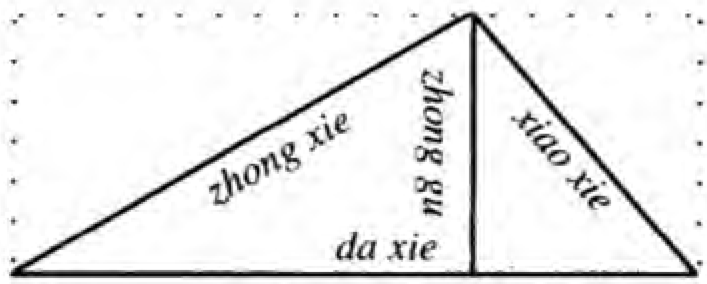
8.

【原文】

今有勾三股四八角田积[1]，以面阔三自乘，加之，却减面阔幂，余二万一千二百八十三步五分步之一。[2] 问：每面阔几何？

答曰：一十二步。

negative *yu*. Solving this expression [4] of the sixth degree we have the required *zhong gu*.



[Notes]

[1] A *san xie tian*, three obliques, is a scalene triangle; the *zhong gu*, middle leg, is the latitude; the *zhong xie*, middle oblique, the mean side; *da xie*, great oblique, the great side; *xiao xie*, small oblique, the small side. Let *san xie tian*'s *da xie* be a, the *zhong xie* b, the *xiao xie* c, the *zhong gu* h, the area S. That is, $S - h = 76$. (G)

[2] That is, $b - h = 9$, $c - h = 2$. (G)

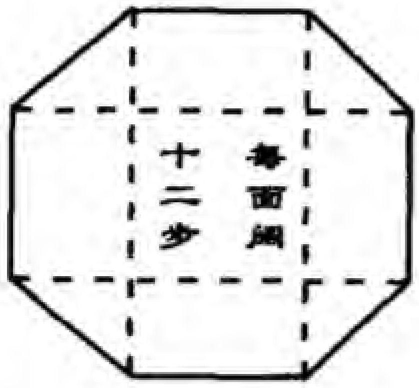
[3] 5371 *bu* 7 *fen* 5 *li* refers to 5371.75 *bu*. The *fen* and *li* following *bu* both refer to decimal fractions. The method of using *fen*, *li*, *hao* and *si* to denote the decimal fraction following an integral unit is derived from *The Master Sun's Mathematical Manual*.(G)

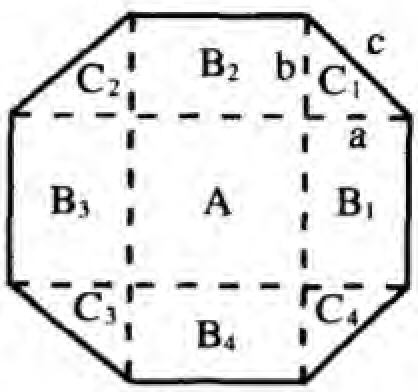
[4] The expression in modern form is the equation: $-49x^6 - 495x^5 + 5371.75x^4 + 278768x^3 + 843296x^2 - 7023616x - 133448704 = 0$. (C)

8. Add to the area of *gou*-3-*gu*-4 octagon [1] the fourth power of its side then subtract from the sum the square of the side; the remainder thus obtained equals $21283\frac{1}{5}$ *bu*.[2] Find a side of the octagon.

Ans. 12 *bu*.

术曰：立天元一为每面之阔，如积求之。得一十万六千四百一十六为益实，一十九为从上廉，五为正隅，三乘方开之，[3] 得每面之阔。合问。





【注释】

[1]一个勾三股四八角田是一个各边相等的八角田。$a:b:c=3:4:5$。（陈）

[2]"三自乘"即自乘三次，亦即四次方。记八角田的面积为S，面阔为x，此即：$S+x^4-x^2=21283\frac{1}{5}$。（郭）

[3]开方式的现代形式为：$5x^4+19x^2-106416=0$。

推导过程如下：

令x=八角田的一边。由以上右面的图可知$a=\frac{4x}{5}$，$b=\frac{3x}{5}$。

A的面积为$A=x^2$

$[B_1+B_2+B_3+B_4]$的面积$=\frac{14x^2}{5}$，$[C_1+C_2+C_3+C_4]$的面积约为x^2

八角田的全面积$=\frac{24x^2}{5}$。

由朱世杰给出的题设可得$\frac{24x^2}{5}+x^4-x^2=21283\frac{1}{5}$

或$x^4+\frac{24x^2}{5}=\frac{106416}{5}$

或$5x^4+19x^2-106416=0$。

$[C_1+C_2+C_3+C_4]$的精确值为$\frac{24x^2}{25}$，且方程应为$5x^4+18.8x^2-106416=0$。（陈）

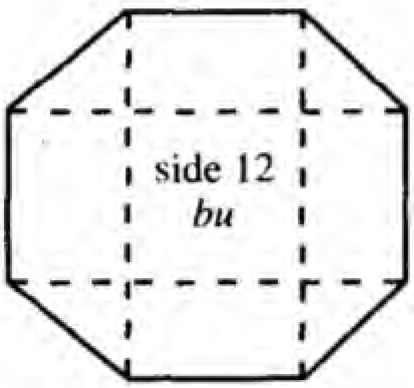
【今译】

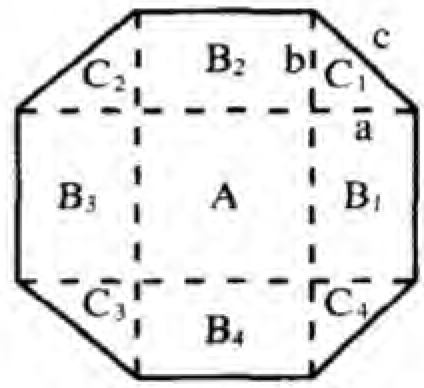
今有勾三股四的八角田的面积加面阔的4次方，减面阔幂，余$21283\frac{1}{5}$步。问：每面阔为多少？

答：12步。

术：设天元一为每面之阔，以如积方法求其解。得到-106416为常数

Process. Let the element *tian* be a side of the octagon. From the statement we have 106416 for the negative *shi*, 19 for the positive first *lian*, and 5 for the positive *yu*, an expression [3] of the fourth degree whose root is the required side of the octagon.





[Notes]

[1] A *gou*-3-*gu*-4 octagon is an octagon which has its sides equal and the ratio of $a : b : c$ as the ratio of 3 : 4 : 5. (C)

[2] The Chinese characters "San Zi Cheng" means that the side of the octagon multiplies itself three times. That is the fourth power of the side. Let the area of the octagon be S, and its side x. That is, $S + x^4 - x^2 = 21283\frac{1}{5}$. (G)

[3] The expression in modern form is the equation: $5x^4 + 19x^2 - 106416 = 0$.

The equation is derived as follows.

Let x = a side of the octagon.

From the figure in the right side on the top, $a = \frac{4x}{5}$, $b = \frac{3x}{5}$.

Area of $A = x^2$

Area of $[B_1 + B_2 + B_3 + B_4] = \frac{14x^2}{5}$

Area of $[C_1 + C_2 + C_3 + C_4] = x^2$, approximately.

Total area of the octagon is $\frac{24x^2}{5}$.

From the author's statement we have

$$\frac{24x^2}{5} + x^4 - x^2 = 21283\frac{1}{5}$$

$$\text{or } x^4 + \frac{24x^2}{5} = \frac{106416}{5}$$

or $5x^4 + 19x^2 - 106416 = 0$.

项，19为二次项系数，5为最高次项系数，开四次方，便得到符合问题的每面之阔。

9.

【原文】

今有三广田积[1]，加中广二分之一，减大广三分之二，又加小广四分之三，减正长六分之五，余，以正长中半乘之，得一万五千八百八十八步。[2] 只云：并三广、正长，虚加二，为实；四为从方，一为从廉，一为从隅，立方开之，并入中广，与小广适等。又开方数如中广三分之一，大小广差二步。[3] 问：长、广各几何？

答曰：小广一十六步，中广一十二步，大广一十八步，正长四十八步。

术曰：立天元一为开方数，如积求之。得一十九万六百五十六为益实，二百三十六为从方，一千九为从上廉，九百八为从二廉，四百二十五为益三廉，二百七十七为益四廉，四十为从下廉，二十一为从隅，六乘方开之，[4] 得四步，为开方数。合问。

【注释】

[1] 三广田是由两个上底相等且相合的梯形组成的多边形。（陈）

[2] 记三广田的面积、大广、小广、中广、正长分别为S，a，b，c，h，此即：$(S+\frac{1}{2}c-\frac{2}{3}a+\frac{3}{4}b-\frac{5}{6}h)\times\frac{1}{2}h=15888$。（郭）

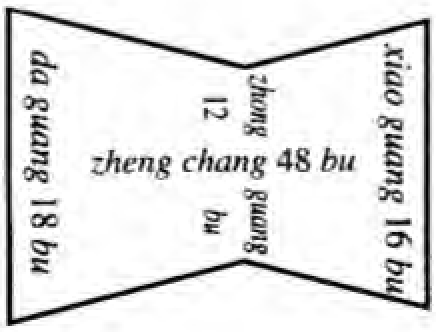
[3] 先求开方式$w^3+w^2+4w=(a+b+c+h+2)$的根w，此即$w+c=b$，$w=\frac{1}{3}c$，$a-b=2$。（郭）

The exact value of $[C_1 + C_2 + C_3 + C_4]$ is $\frac{24x^2}{25}$ and the equation ought to be $5x^4 + 18.8x^2 - 106416 = 0$. (C)

9. The area of a *san guang tian* [1] plus one-half of the *zhong guang*, minus two-thirds of the *da guang*, plus three-fourths of the *xiao guang*, minus five-sixths of the *zheng chang*, and the result multiplied by one-half of the *zheng chang* is equal to 15888 *bu*.[2] If we take the sum of the three *guang*, the *zheng chang*, and 2 for the positive *shi*, 4 for the positive *fang*, 1 for the positive *lian*, and 1 for the positive *yu* of a cubic expression, its root with the *zhong guang* is equal to the *xiao guang*. The root is also equal to one-third of the *zhong guang*. The difference between the *da guang* and the *xiao guang* is 2 *bu*.[3] Find all the *guang* and the *chang*.

Ans. *Xiao guang*, 16 *bu*; *zhong guang*, 12 *bu*; *da guang*, 18 *bu*; *zheng chang*, 48 *bu*.

Process. Let the element *tian* be the root of the expression. From the statement we have 190656 for the negative *shi*, 236 for the positive *fang*, 1009 for the positive first *lian*, 908 for the positive second *lian*, 425 for the negative third *lian*, 277 for the negative fourth *lian*, 40 for the positive last *lian*, and 21 for the positive *yu*, an expression[4] of the seventh degree whose root, 4 *bu*, is the required root.



[4] 开方式的现代形式为：$21x^7+40x^6-277x^5-425x^4+908x^3+1009x^2+236x-190656=0$。（陈）

【今译】

今有三广田的面积，加中广的$\frac{1}{2}$，减大广的$\frac{2}{3}$，加小广的$\frac{3}{4}$，减正长的$\frac{5}{6}$，其余数乘以正长的$\frac{1}{2}$，得15888步。只云：以大广、中广、小广、正长与2的和为常数项，4为一次项系数，1为二次项系数，1为最高次项系数，开立方。其根与中广之和等于小广，又，其根与中广的$\frac{1}{3}$相等，大广与小广之差为2步。问：大广、中广、小广、正长各为多少？

答：小广16步，中广12步，大广18步，正长48步。

术：设天元一为开方数，以如积方法求其解。得到-190656为常数项，236为一次项系数，1009为二次项系数，908为三次项系数，-425为四次项系数，-277为五次项系数，40为六次项系数，21为最高次项系数，开七次方，得到4步，便是开方数。符合所问。

10.

【原文】

今有种金田积[1]，加对尖直长，得三百一十五步。[2] 只云外两斜各长二十五步，内两斜各长二十步。[3] 问：对尖直长几何？

答曰：一十五步。

术曰：立天元一为对尖直长，如积求之。得四十四万七千五百二十五为益实，二千五百二十为从方，二千四十六为从上廉，一为益隅，三乘方开之，[4] 得对尖直长。合问。

[Notes]

[1] A *san guang tian*(land with three widths)is polygon formed by two trapezoids having their upper bases equal and coincident. (C)

[2]Let the *san guang tian*'s area, *da guang*, *xiao guang*, *zhong guang*, *zheng chang* be S, a, b, c, h. That is, $(S + \frac{1}{2}c - \frac{2}{3}a + \frac{3}{4}b - \frac{5}{6}h) \times \frac{1}{2}h = 15888$. (G)

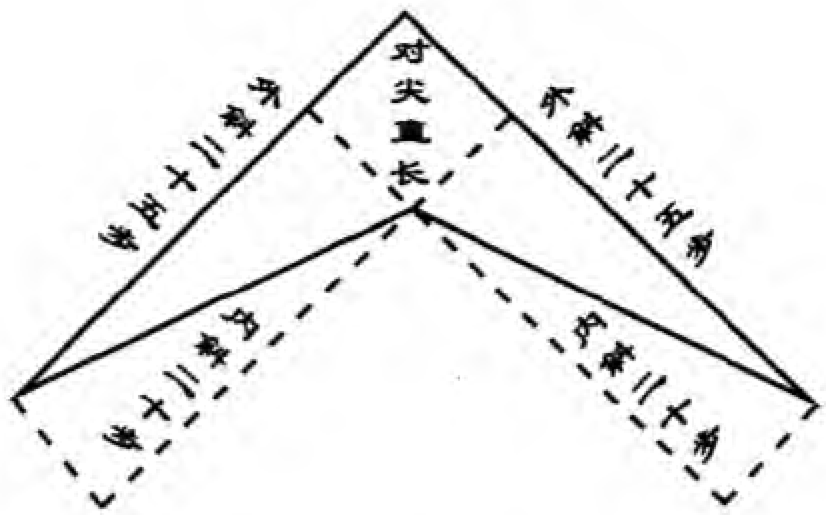
[3] First, solve the root w of the equation $w^3 + w^2 + 4w = (a + b + c + h + 2)$. That is, $w + c = b$, $w = \frac{1}{3}c$, $a - b = 2$. (G)

[4] The expression in modern form is the equation: $21x^7 + 40x^6 - 277x^5 - 425x^4 + 908x^3 + 1009x^2 + 236x - 190656 = 0$. (C)

10.The area of a *zhong jin tian* [1] plus the diagonal joining the two vertices is equal to 315 *bu*.[2] It is said that each of the two *wai xie* equals 25 *bu*, and each of the *nei xie* 20 *bu*.[3] Find the diagonal.

Ans. 15 *bu*.

Process. Let the element *tian* be the diagonal. From the statement we have 447525 for the negative *shi*, 2520 for the positive *fang*, 2046 for the positive first *lian*, and 1 for the negative *yu*, an expression [4] of the fourth degree whose root is the required diagonal.



【注释】

［1］种金田形如金字的顶部，如图所示为一多边形。（陈）

［2］记种金田的面积及对尖直长分别为 S，l，此即：$S + l = 315$。（郭）

［3］记外两斜为 a，内两斜为 b，此即：$a = 25$，$b = 20$。（郭）

［4］开方式的现代形式为：$-x^4 + 2046x^2 + 2520x - 447525 = 0$。（陈）

【今译】

今有种金田的面积加对尖直长，得315步。只云外两斜各长25步，内两斜各长20步。问：对尖直长为多少？

答：15步。

术：设天元一为对尖直长，以如积方法求其解。得到−447525为常数项，2520为一次项系数，2046为二次项系数，−1为最高次项系数，开四次方，便得到对尖直长。符合所问。

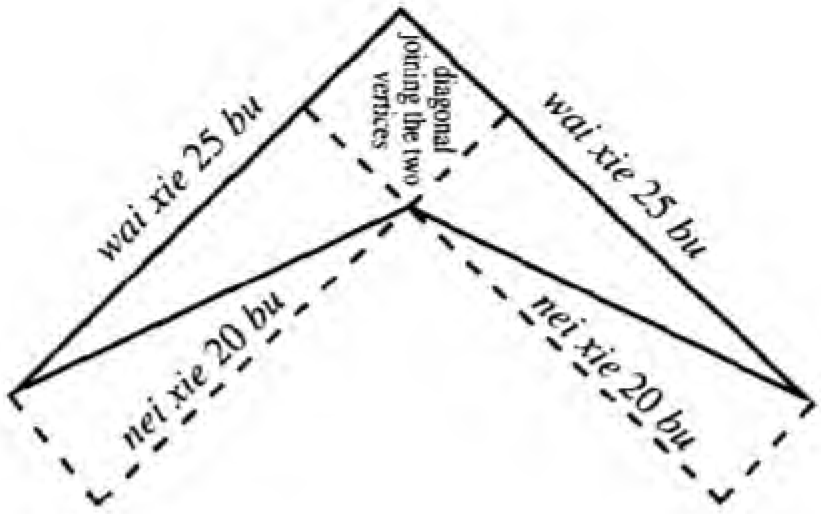
11.

【原文】

今有圆田积，加圆径，减圆周，余自乘，加径幂，得七千二百步。[1] 问：圆径几何？

答曰：一十二步。

术曰：立天元一为圆径，如积求之。得一十一万五千二百为益实，八



[Notes]

[1] A *zhong jin tian* [formed like the top of the character *jin* (金) gold] is polygon shown by the figure above. (C)

[2] Let the *zhong jin tian*' s area and the diagonal be S and l. That is, $S + l = 315$. (G)

[3] Let two *wai xie* be a, and two *nei xie* b. That is, $a = 25, b = 20$. (G)

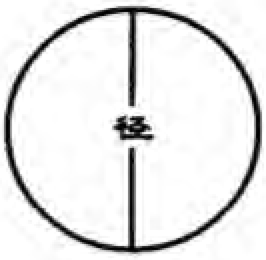
[4] The expression in modern form is the equation: $-x^4 + 2046x^2 + 2520x - 447525 = 0$. (C)

11. To the square of the diameter of a circle add the square of the area, plus the diameter, minus the circumference. The result equals 7200 *bu*.[1] Find the dameter.

Ans. 12 *bu*.

Process. Let the element *tian* be the diameter of the circle. From the statement we have 115200 for the negative *shi*, 80 for the positive first *lian*, 48

十为从上廉，四十八为益下廉，九为正隅，三乘方开之[2]，得圆径。合问。



【注释】

[1] 圆田面积、圆周、圆径分别为 S，l，d。此即：$(S+d-l)^2+d^2=7200$。（郭）

[2] 开方式的现代形式为：$9x^4-48x^3+80x^2-115200=0$。（陈）

【今译】

今有圆田的面积加圆径，减圆周，余数自乘，加径幂，得7200步。问：圆径为多少？

答：12步。

术：设天元一为圆径，以如积方法求其解。得到 −115200 为常数项，80为二次项系数，−48为三次项系数，9为最高次项系数，开四次方，便得到圆径。符合所问。

12.

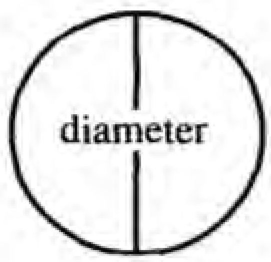
【原文】

今有窊田积[1]，加窊径强半，减上周太半，余一百三十二步。[2] 只云：窊径，以平方开之，并入上周，共得四十步。[3] 问：周、径各几何？

答曰：周三十六步，径一十六步。

术曰：立天元一为上周，如积求之。得一万二千八百一十六为正实，四千七十二为从方，三百二十一为益廉，三为正隅，立方开之，[4] 得上周。合问。

for the negative last *lian*, and 9 for the positive *yu*, an expression[2] of the fourth degree whose root is the required diameter.



[Notes]

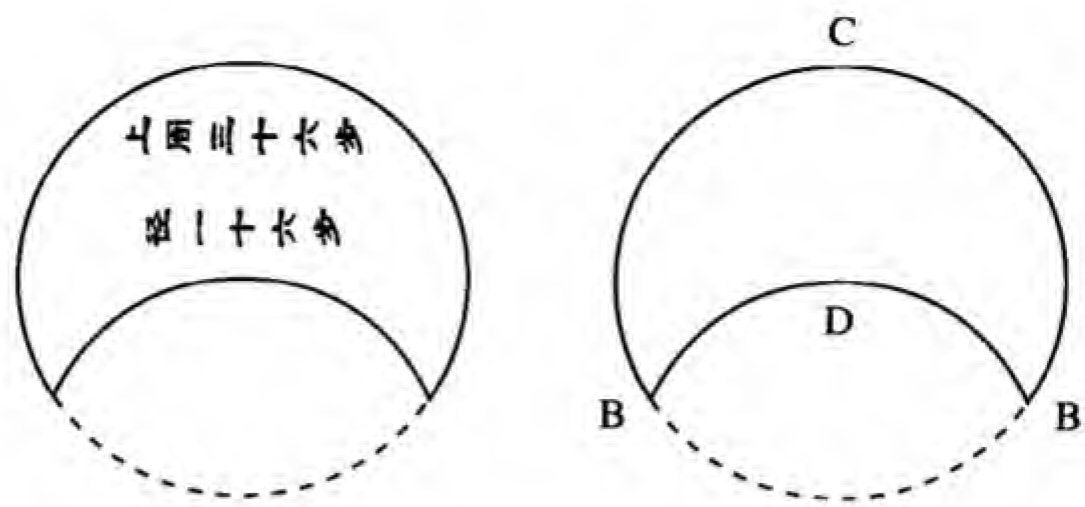
[1] Let the circle's area, the circumference, and the diameter be S, l, and d. That is, $(S + d - l)^2 + d^2 = 7200$. (G)

[2] The expression in modern form is the equation: $9x^4 - 48x^3 + 80x^2 - 115200 = 0$. (C)

12. The area of a *wa tian*[1] plus the strong half of the *wa jing* minus the great half of the *shang zhou* equals 132 *bu*;[2] and the sum of the square root of the *wa jing* and the *shang zhou* equals 40 *bu*.[3] Find the *shang zhou* and the *wa jing*.

Ans. *Shang zhou*, 36 *bu*; *wa jing*, 16 *bu*.

Process. Let the element *tian* be the *shang zhou*. From the statement we have 12816 for the positive *shi*, 4072 for the positive *fang*, 321 for the negative *lian*, and 3 for the positive *yu*, a cubic expression[4] whose root is the required *shang zhou*.



【注释】

[1] 窊田如图中所示为圆的一部分。$\widehat{ACB}$ 称作上周，AB 为窊径。窊田面积$=\frac{1}{4}\widehat{ACB}\cdot AB$。（陈）

[2] 窊，音 wa，低凹，低下。窊田是一种优半球形状的田。强半为$\frac{3}{4}$。记窊田的面积、窊径、上周分别为 S，d，l，此即：$S+\frac{3}{4}d-\frac{2}{3}l=132$。（郭）

[3] 此即：$\sqrt{d}+l=40$。（郭）

[4] 开方式的现代形式为：$3x^3-321x^2+4072x+12816=0$。（陈）

【今译】

今有一窊田，其面积加窊径的$\frac{3}{4}$，减上周的$\frac{2}{3}$，余 132 步。只云：将窊径开平方，加入上周，共得 40 步。问：窊田的周、径各为多少？

答：周 36 步，径 16 步。

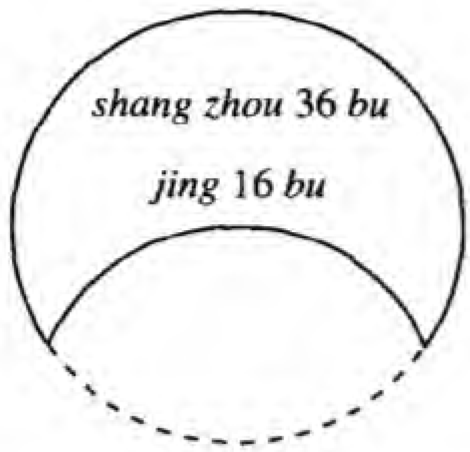
术：设天元一为上周，以如积方法求其解。得到 12816 为常数项，4072 为一次项系数，−321 为二次项系数，3 为最高次项系数，开三次方，便得到上周。符合所问。

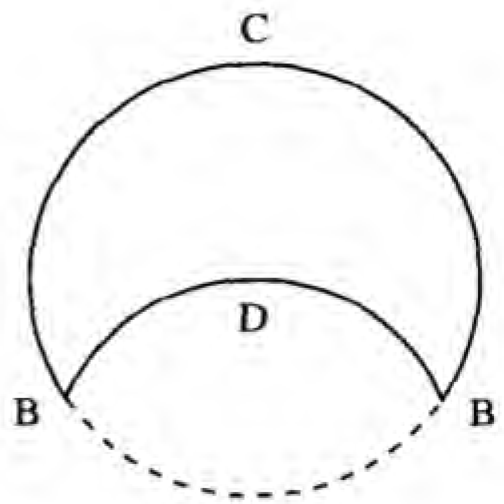
13.

【原文】

今有畹田积[1]，加下周幂少半，减畹径幂太半，余二千七百九十五步弱半步。[2] 只云：下周为实，二为从方，一为从隅，平方开之；又畹径减二，余以平方开之，少如先开方数二步。[3] 问：周、径几何？

答曰：周九十九步，径五十一步。





[Notes]

[1] A *wa tian*, depressed circle, is a section of a circle as shown in the figure. The $\widehat{ACB}$ is called the *shang zhou* (upper circumference) and the AB the *wa jing* (depressed diameter). The area is $\frac{1}{4}\widehat{ACB} \cdot AB$. (C)

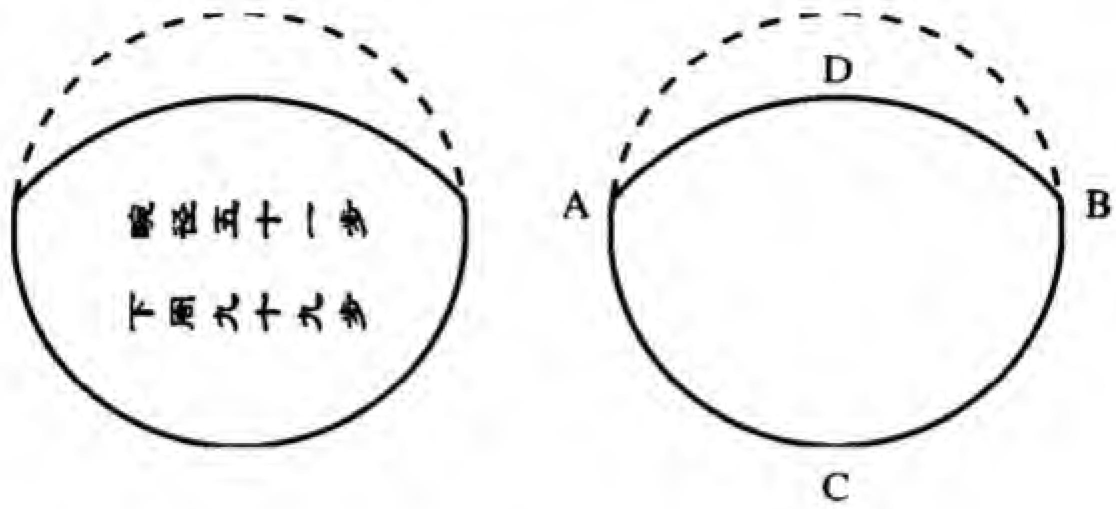
[2] The character *wa* means concave in Chinese. *Wa tian* is a big section of a circle. A great half is $\frac{3}{4}$. Let the *wa tian*'s area, the *wa jing*, and the *shang zhou* be S, d, and l. That is, $S + \frac{3}{4}d - \frac{2}{3}l = 132$. (G)

[3] That is, $\sqrt{d} + l = 40$. (G)

[4] The expression in modern form is the equation: $3x^3 - 321x^2 + 4072x + 12816 = 0$. (C)

13. The area of a *wan tian*[1] plus the small half of the square of the *xia zhou*, minus the great half of the square of the *wan jing* equals 2795 *bu* less one-half *bu*.[2] It is said that if the *xia zhou* be put for the *shi*, 2 for the positive *fang*, and 1 for the positive *yu* of a quadratic expression, the square root of the difference between the *wan jing* and 2 is less than the root of this quadratic

术曰：立天元一为先开方数，如积求之。得三万三千八百三十一为益实，四百二十为从方，二百一十四为益上廉，七十四为从下廉，一为益隅，三乘方开之，[4] 得九步，为先开方数。合问。



【注释】

[1] 畹田为球冠形。设下周为 $\widehat{ACB}$，畹径为 AB，则畹田面积 $= \frac{1}{4}\widehat{ACB}\cdot AB$。（陈）

[2] 少半即 $\frac{1}{3}$。弱半为 $\frac{1}{4}$。记畹田面积、下周、畹径分别为 S，l，d，《九章算术》提出面积公式：$S = \frac{1}{4}dl$。此即：$S + \frac{1}{3}l^2 - \frac{2}{3}d^2 = 2795\frac{1}{4}$。（郭）

[3] 先求开方式 $x^2 + 2x = l$ 的根 x，为先开方数。$x - \sqrt{d-2} = 2$。（郭）

[4] 开方式的现代形式为：$-x^4 + 74x^3 - 214x^2 + 420x - 33831 = 0$。（陈）

【今译】

今有畹田的面积加下周幂的 $\frac{1}{3}$，减畹径幂的 $\frac{2}{3}$，余 $2795\frac{1}{4}$ 步。只云：以下周为实，2 为一次项系数，1 为最高次项系数，开平方；又，将畹径减 2，开平方，其根比先开方得数少 2 步。问：畹田的周、径各为多少？

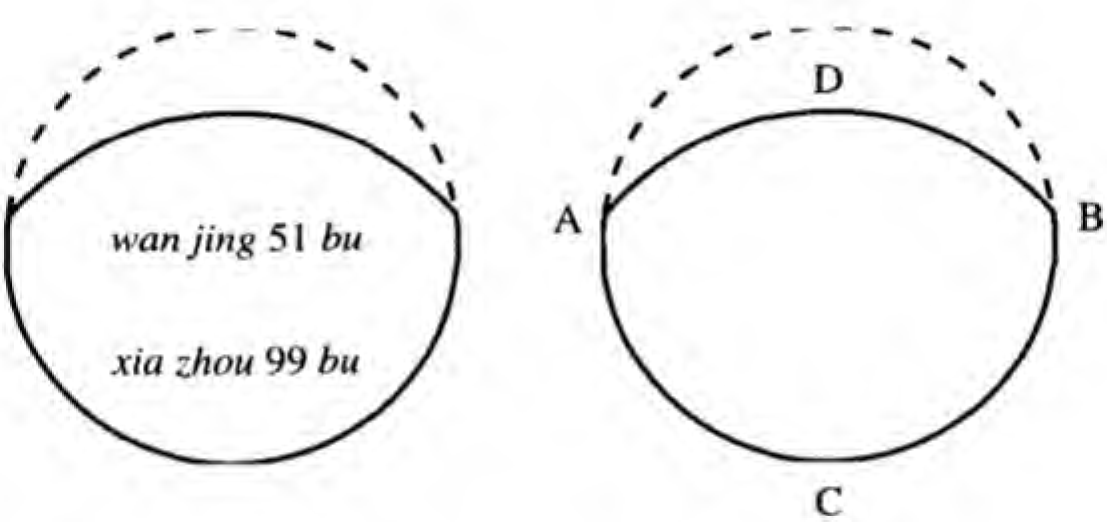
答：周 99 步，径 51 步。

术：设天元一为先开方得数，以如积方法求其解。得到 −33831 为常数项，420 为一次项系数，−214 为二次项系数，74 为三次项系数，−1 为最高次项系数，开四次方，便得到 9 步，为先开方得数。符合所问。

expression by 2 *bu*.[3] Find the *zhou* and the *jing*.

Ans. Zhou, 99 *bu*; jing, 51 *bu*.

Process. Let the element *tian* be the root of the quadratic expression. From the statement we have 33831 for the negative *shi*, 420 for the positive *fang*, 214 for the negative first *lian*, 74 for the positive last *lian*, and 1 for the negative *yu*, an expression[4] of the fourth degree with 9 as the required root.



[Notes]

[1] A *wan tian* is a section of a sphere as shown in the figure formed by cutting a curved line convex upward; the *xia zhou* is the $\overset{\frown}{ACB}$; the wan *jing*(diameter of the convex curve) is AB. The area is $\frac{1}{4}\overset{\frown}{ACB}\cdot AB$. (C)

[2] A small half is $\frac{1}{3}$. A less one-half is $\frac{1}{4}$. Let the *wan tian*'s area, the *xia zhou*, and the *wan jing* be S, l, and d. The formula of area was given in *The Nine Chapters of Mathematical Procedures*: $S = \frac{1}{4}dl$. That is, $S + \frac{1}{3}l^2 - \frac{2}{3}d^2 = 2795\frac{1}{4}$. (G)

[3] First, extract the root x of the equation $x^2 + 2x = l$ as *xian kai fang shu*. $x - \sqrt{d-2} = 2$. (G)

[4] The expression in modern form is the equation: $-x^4 + 74x^3 - 214x^2 + 420x - 33831 = 0$. (C)

14.

【原文】

今有球露钱田积，加面径，减圆周，余五十六步。[1] 只云虚径多如面径二步[2]。问：三径各几何？

答曰：面径四步，虚径六步，通径一十四步。

术曰：立天元一为面径，如积求之。得一百二十八为益实，一十二为益方，一十一为正隅，平方开之，[3] 得面径。合问。

【注释】

[1] 球露钱田是一圆内由四段相等的圆弧割去中间的部分所形成的田。记其面积、通径、面径、圆周分别为 S，d，r，l，则虚径为 $d-2r$。此即：$S+r-l=56$。（郭）

[2] 此即：$(d-2r)-r=2$。（郭）

[3] 开方式的现代形式为：$11x^2-12x-128=0$。（陈）

【今译】

今有球露钱田的面积加面径，减圆周，余56步。只云虚径比面径多2步。问：面径、虚径、通径各为多少？

答：面径4步，虚径6步，通径14步。

术：设天元一为面径，以如积方法求其解。得到 −128 为常数项，−12 为一次项系数，11 为最高次项系数，开平方，便得到面径。符合所问。

14. The area of a *qiu lu qian tian* plus the *mian jing*, minus the circumference of the circle equals 56 *bu*.[1] The *xu jing* exceeds the *mian jing* by 2 *bu*[2]. Find the three *jing*.

Ans. *Mian jing*, 4 *bu*;

xu jing, 6 *bu*;

tong jing, 14 *bu*.

Process. Let the element *tian* be the *mian jing*. From the statement we have 128 for the negative *shi*, 12 for the negative *fang*, and 11 for the positive *yu*, a quadratic expression[3] whose root is the required *mian jing*.

【Notes】

[1] A *qiu lu qian tian* is formed by four equal arc and the circumference of a circle in which the middle part is cut off. Let its area, *tong jing*, *mian jing*, and circumference be S, d, r, l. Then, *xu jing* is $d - 2r$. That is, $S + r - l = 56$. (G)

[2] That is, $(d - 2r) - r = 2$. (G)

[3] The expression in modern form is the equation: $11x^2 - 12x - 128 = 0$. (C)

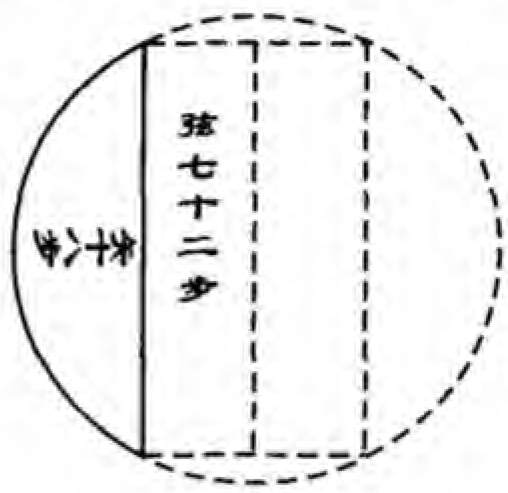
15.

【原文】

今有弧田积，加矢立幂，减弦平幂，余，以矢除之，加矢立幂，得五千九百一十三步。[1] 只云矢除弦，得四步。[2] 问：弦、矢各几何?

答曰：矢一十八步，弦七十二步。

术曰：立天元一为矢，如积求之。得一万一千八百二十六为益实，二十七为益方，二为从廉，二为正隅，立方开之，[3] 得矢。合问。



【注释】

[1] 弧田即今之弓形，记其面积、弦、矢分别为 S，c，v，《九章算术》方田章给出其面积公式：$S=\frac{1}{2}(cv+v^2)$。此条题设是：$\frac{S+v^3-c^2}{v}+v^3=5913$。（郭）

[2] 此即：$\frac{c}{v}=4$。（郭）

[3] 开方式的现代形式为：$2x^3+2x^2-27x-11826=0$。（陈）

【今译】

今有弧田的面积加矢的立方，减弦的平方，其余数除以矢，再加矢的立方，得 5913 步。只云以矢除弦，得 4 步。问：弦、矢各为多少?

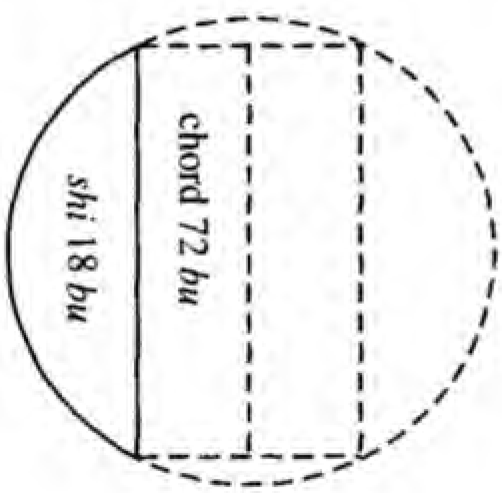
答：矢 18 步，弦 72 步。

术：设天元一为矢，以如积方法求其解。得到 -11826 为常数项，-27 为一次项系数，2 为二次项系数，2 为最高次项系数，开三次方，便得到矢。符合所问。

15. To the area of a segment of a circle add the cube of the *shi*, subtract the square of the subtended chord, divide the remainder by the *shi* and add to the quotient the cube of the *shi*. The result equals 5913 *bu*.[1] It is said that the quotient of the chord and the *shi* is equal to 4 *bu*.[2] Find the chord and the *shi*.

Ans. *Shi*, 18 *bu*; chord, 72 *bu*.

Process. Let the element *tian* be the *shi*. From the statement we have 11826 for the negative *shi*, 27 for the negative *fang*, 2 for the positive *lian*, and 2 for the positive *yu*, a cubic expression[3] whose root is the required *shi*.



【Notes】

[1] A *hu tian* is a segment of a circle, which like a bow. Let its area, chord, and *shi* be S, c, and v. The formula of area was given by the chapter of *fang tian* (surveying of land) of *The Nine Chapters of Mathematical Procedures*: $S = \frac{1}{2}(cv + v^2)$. The statement in the problem is $\frac{S + v^3 - c^2}{v} + v^3 = 5913$. (G)

[2] That is, $\frac{c}{v} = 4$. (G)

[3] The expression in modern form is the equation: $2x^3 + 2x^2 - 27x - 11826 = 0$. (C)

16.

【原文】

今有车辋田积，以径乘内周加之，以外周乘径减之，又以径幂乘之，减径幂，余三千五百二十八步。[1] 只云径幂多如外周六步，内外周差九步。[2] 问：实径及内外周各几何？

答曰：实径六步，内周二十一步，外周三十步。

术曰：立天元一为辋径，如积求之。得三千五百二十八为益实，一为益上廉，一十九步半为益二廉，一为正隅，四乘方开之，[3] 得实径。合问。

【注释】

[1] 车辋田即一段圆环的形状，《九章算术》已有其例题，并用环田术求解。记环田的面积、径、内周、外周为 S，d，l_1，l_2，《九章算术》给出的环田术是：$S=\frac{1}{2}(l_1+l_2)d$。此条题设就是：$(S+dl_1-dl_2)d^2-d^2=3528$。（郭）

[2] 此即：$d^2-l_2=6$，$l_2-l_1=9$。（郭）

[3] 开方式的现代形式为：$x^5-19\frac{1}{2}x^3-x^2-3528=0$。（陈）

【今译】

今有车辋田的面积，加上以实径乘内周，减去以实径乘外周，再以实径幂乘之，减去实径幂，余3528步。只云实径幂比外周多6步，内外周之差为9步。问：实径及内外周各为多少？

答：实径6步，内周21步，外周30步。

术：设天元一为车辋田的实径，以如积方法求其解。得到 −3528 为常数项，−1 为二次项系数，$-19\frac{1}{2}$ 为三次项系数，1 为最高次项系数，开五次方，便得到实径。符合所问。

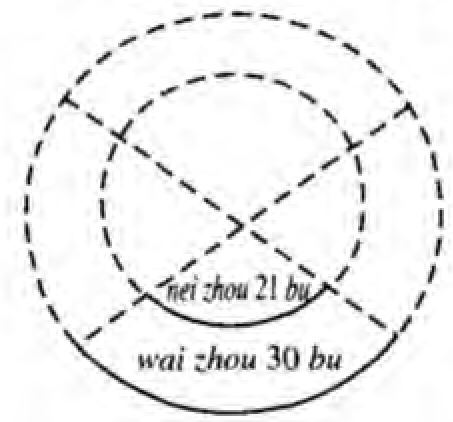
16. The area of the *che wang tian* plus the product of the *jing* by the *nei zhou*, minus the product of the *wai zhou* by the *jing*, and the remainder multiplied by the square of the *jing*. The result minus the square of the *jing* equals 3528 *bu*.[1] It is said that the square of the *jing* exceeds the *wai zhou* by 6 *bu*, and the difference between the *wai zhou* and the *nei zhou* is 9 *bu*.[2] Find the *shi jing*, the *nei zhou*, and the *wai zhou*.

Ans. *Shi jing*, 6 *bu*;

nei zhou, 21 *bu*;

wai zhou, 30 *bu*.

Process. Let the element *tian* be the *jing* of the *che wang tian*. From the statement we have 3528 for the negative *shi*, 1 for the negative first *lian*, $19\frac{1}{2}$ for the negative second *lian*, and 1 for the positive *yu*, an expression[3] of the fifth degree whose root is the required *shi jing*.



[Notes]

[1] A *che wang tian* is a segment of a wheel. There are examples in *The Nine Chapters of Mathematical Procedures*, which were solved by the *huan tian* method. Let the *huan tian*'s area, *jing*, *nei zhou*, and *wai zhou* be S, d, l_1, l_2. The *huan tian* method given in *The Nine Chapters of Mathematical Procedures* is as follows: $S = \frac{1}{2}(l_1 + l_2) \cdot d$. The statement of the problem is $(S + dl_1 - dl_2)d^2 - d^2 = 3528$. (G)

[2] That is, $d^2 - l_2 = 6$, $l_2 - l_1 = 9$. (G)

[3] The expression in modern form is the equation: $x^5 - 19\frac{1}{2}x^3 - x^2 - 3528 = 0$. (C)

17.

【原文】

今有钱田积幂，加一池方面，减四钱田积，余一十二万一千八百一十五步。[1] 只云博径三步。[2] 问：池方、田周各几何？

答曰：池方一十八步，田周九十步。

术曰：立天元一为池方，如积求之。得四十八万四千七百七十六为益实，一千八百四为从方，四百二十四为从上廉，三十六为从下廉，一为正隅，三乘方开之，[3] 得池方。合问。

【注释】

[1] 钱田是圆田中间挖去一正方形水池的形状。记钱田的面积、水池的边长分别为 S，a，此即：$S^2+a-4S=121815$。（郭）

[2] 博径为水池边长的延长线自水池的顶点到圆周的距离，为 3 步。（郭）

[3] 开方式的现代形式为：$x^4+36x^3+424x^2+1804x-484776=0$。（陈）

【今译】

今有一钱田的面积之幂，加中间水池的一边长，减去钱田面积的 4 倍，余 121815 步。只云博径为 3 步。问：水池的一边长、钱田的周长各为多少？

答：水池的一边长 18 步，钱田的周长 90 步。

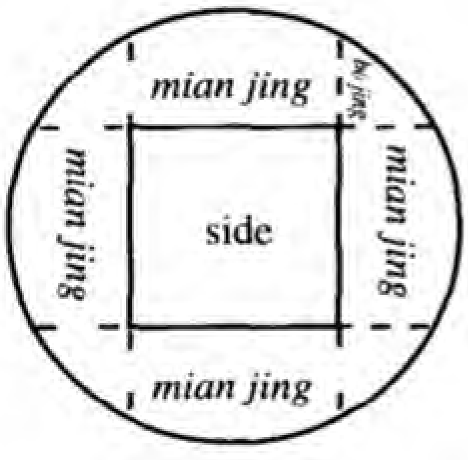
术：设天元一为水池的一边长，以如积方法求其解。得到 −484776 为常数项，1804 为一次项系数，424 为二次项系数，36 为三次项系数，1 为最高次项系数，开四次方，便得到水池一边长。符合所问。

17. The square of the area of a *qian tian* plus a side of the square pool in the centre of the land, minus 4 times the area of *qian tian* equals 121815 *bu*.[1] It is said that the *bo jing* is 3 *bu*.[2] Find a side of the pool and the circumference of the land.

Ans. A side of the pool, 18 *bu*;

circumference of the land, 90 *bu*.

Process. Let the element *tian* be a side of the pool. From the statement we have 484776 for the negative *shi*, 1804 for the positive *fang*, 424 for the positive first *lian*, 36 for the positive last *lian*, and 1 for the positive *yu*, an expression[3] of the fourth degree whose root is the required side of the pool.



[Notes]

[1] A *qian tian* is a form that the middle of a *yuan tian* is dug a square pool just as a cash form. Let the *qian tian*'s area and a side of the pool be S and a. That is, $S^2 + a - 4S = 121815$. (G)

[2] A *bo jing* is the distance from the top of the pool to the circumference of the circle, which is 3 *bu*. (G)

[3] The expression in modern form is the equation: $x^4 + 36x^3 + 424x^2 + 1804x - 484776 = 0$. (C)

18.

【原文】

今有环田积，实径乘外周加之，却减内周幂，余七百二十九步。[1] 只云：并内外周，减二，余，以平方开之，所得不及实径一步。[2] 问：周、径各几何？

答曰：实径一十五步，内周五十四步，外周一百四十四步。

术曰：立天元一为实径，如积求之。得二千九百二十五为益实，六十为从方，六十六为益上廉，二十为从下廉，一为益隅，三乘方开之，[3] 得实径。合问。

【注释】

[1] 记环田的的面积、实径、内周、外周为 S, d, l_1, l_2,《九章算术》给出的环田术是：$S=\frac{1}{2}(l_1+l_2)d$。此条题设即：$S+dl_2-l_1^2=729$。（郭）

[2] 此即：$d-\sqrt{l_1+l_2-2}=1$。（郭）

[3] 开方式的现代形式为：$-x^4+20x^3-66x^2+60x-2925=0$。（陈）

【今译】

今有环田的面积加上实径乘外周，减去内周之幂，余729步。只云：内外周相加，减去2，其余数开平方，其根小于实径1步。问：内外周、实径各为多少？

答：实径15步，内周54步，外周144步。

术：设天元一为实径，以如积方法求其解。得到 −2925 为常数项，60 为一次项系数，−66 为二次项系数，20 为三次项系数，−1 为最高次项系数，开四次方，便得到实径。符合所问。

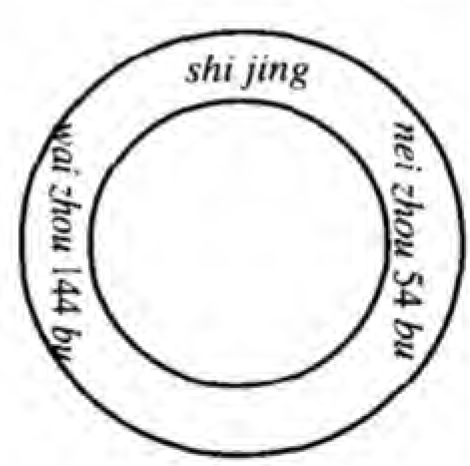
18. Add to the area of a *huan tian* the product of the *shi jing* by the *wai zhou* and subtract from the sum of the square of the *nei zhou*. The remainder is 729 *bu*.[1] It is said that the square root of the sum of the *nei zhou* and the *wai zhou* less 2 is less than the *shi jing* by 1 *bu*.[2] Find the *nei zhou*, *wai zhou*, and *shi jing*.

Ans. *Shi jing*, 15 *bu*;

nei zhou, 54 *bu*;

wai zhou, 144 *bu*.

Process. Let the element *tian* be the *shi jing*. From the statement we have 2925 for the negative *shi*, 60 for the positive *fang*, 66 for the negative first *lian*, 20 for the positive last *lian*, and 1 for the negative *yu*, an expression[3] of the fourth degree whose root is the required *shi jing*.



【Notes】

[1] Let the *huan tian*'s area, *shi jing*, *nei zhou*, and *wai zhou* be S, d, l_1, l_2. The *huan tian* method given in *The Nine Chapters of Mathematical Procedures* is as follows: $S = \frac{1}{2}(l_1 + l_2)d$. The statement of the problem is $S + dl_2 - l_1^2 = 729$. (G)

[2] That is, $d - \sqrt{l_1 + l_2 - 2} = 1$. (G)

[3] The expression in modern form is the equation: $-x^4 + 20x^3 - 66x^2 + 60x - 2925 = 0$. (C)

端匹互隐 九问

1.

【原文】

今有钱三贯四百一十九文，买罗一端。[1] 只云端长内加八尺之价，共得五百七十八文尺。[2] 问：端长、尺价各几何？

答曰：端长五丈二尺，尺价六十五文四分文之三。

术曰：立天元一为尺价，如积求之。得三千四百一十九为益实，五百七十八为从方，八为益隅，平方开之，[3] 得尺价。不尽，以连枝同体术[4] 求之。合问。

【注释】

[1] 1贯为1000文。设天元一x为尺价，此即$\frac{3419}{x}$ = 端尺数。(郭)

[2] 此即：端尺数 + $8x = 578$。(郭)

[3] 开方式的现代形式为：$-8x^2+578x-3419=0$。(陈)

[4] 连枝同体术，又称为之分法。当求出开方式（设为n次）的根的整数部分后，发现开方不尽，便以减根开方式（即求出根的整数部分之后的余式）的隅（开方式的最高次方的系数）的$n-1$，$n-2$，……1，0，-1次方乘其常数项，及1，2，……，$n-2$，$n-1$，n次项的系数，使n次项的系数变成1。记减根开方式的最高次方的系数为A，这相当于进行变换$y=Ax$。开方求出y，则原开方式的分数部分就是$x_1=\frac{y}{A}$。(郭）以连枝同体术解此表达式的现代符号表述形式为：

Duan Pi Hu Yin (Problems on Piece Goods)

9 Problems

1. A *duan* of silk costs 3419 cash.[1] If a number corresponding to the price of 8 *chi* of silk be added to the number of *chi* in the duan the sum is 578 *wen chi*.[2] Find the length of the *duan* and the price of the *chi*.

Ans. Length, 5 *zhang* 2 *chi*;

price per *chi*, $65\frac{3}{4}$ cash.

Process. Let the element *tian* be the price of one *chi*. From the statement we have 3419 for the negative *shi*, 578 for the positive *fang*, and 8 for the negative *yu*, a quadratic expression.[3] Since the root of this expression is not an integral number we apply, in solving, the *lian zhi tong ti* method[4].

[Notes]

[1] One *guan* is 1000 cash. Let the element *tian* be the price of *chi*. Use x for the *tian*. That is, $\frac{3419}{x}$ = the number of *duan*. (G)

[2] That is, the length of *duan* $+ 8x = 578$. (G)

[3] The expression in modern form is the equation: $-8x^2 + 578x - 3419 = 0$. (C)

[4] The *lian zhi tong ti* method is also called *zhi fen* method. When one gets the integral part of the root of a equation (let its power be n), he finds the extraction never-ending. He multiplies the constant term by *yu*'s powers $n-1, n-2, \cdots\cdots 1, 0, -1$ in *jian gen kai fang shi* (the remainder after extracting the integral part of the root), the coefficients of the terms of the powers $1, 2, \cdots\cdots n-2, n-1, n$, and makes the coefficient of the term of the power n become 1.Let the coefficient of the term of the highest power of *jian gen kai fang shi* be A. This is the same as the commutation $y = Ax$. By extraction he gets y, then the fractional part of the root of the original equation is $x_1 = \frac{y}{A}$. (G)Solving this expression by the *lian zhi tong ti* method, in modern symbols, we have

−8	+578	−3419	（60
	−480	+5880	5
	98	+2461	65
	−480	−2116	
	−382	+355	
	−40		
	−422		
	−8		
	−430		

因此$65\frac{321}{430}$或者$65\frac{3}{4}$为所求之根。（陈）

【今译】

今有钱3贯419文，买罗1端。只云端长加8尺的价钱，共得578文尺。问：端长、尺价各为多少？

答：端长5丈2尺，尺价$65\frac{3}{4}$文。

术：设天元一为尺价，以如积方法求其解。得到−3419为常数项，578为一次项系数，−8为最高次项系数，开平方，便得到尺价。开方不尽，以连枝同体术求之，便符合所问。

2.

【原文】

今有绫一匹，直钱一贯五百四十八文。只云尺价内减匹长，余，以尺价乘之，减尺价，余一贯三百一十四文。[1] 问：匹长、尺价各几何？

答曰：匹长二丈八尺三分尺之二，尺价五十四文。

术曰：立天元一为尺价，如积求之。得二千八百六十二为益实，一为益方，一为正隅，平方开之，[2] 得尺价。合问。

–8	+578	–3419	(60
	–480	+5880	5
	98	+2461	65
	–480	–2116	
	–382	+355	
	–40		
	–422		
	–8		
	–430		

Therefore, $65\frac{321}{430}$ or $65\frac{3}{4}$ is the required root. (C)

2. The price of a *pi* of damask is 1548 cash. It is said that if we subtract the number of *chi* in the *pi* from the price of one *chi*, multiply the remainder by the price of one *chi*, then subtract from the product the price of one *chi*, the remainder thus obtained is 1314.[11] Find the length of the *pi* and the price of one *chi*.

Ans. Length, 2 *zhang* $8\frac{2}{3}$ *chi*;

【注释】

[1] 设天元一 x 为尺价，记 w 为匹长，此即：$(x-w)x-x=1314$。（郭）

[2] 开方式的现代形式为：$x^2-x-2862=0$。（陈）

【今译】

今有绫1匹，值钱1贯548文。只云尺价中减匹长，余，以尺价乘之，减去尺价，余1贯314文。问：匹长、尺价各为多少？

答：匹长2丈$8\frac{2}{3}$尺，尺价54文。

术：设天元一为尺价，以如积方法求其解。得到−2862为常数项，−1为一次项系数，1为最高次项系数，开平方，便得到尺价。符合所问。

3.

【原文】

今有锦一端，直钱四贯八十文。[1] 只云并尺价、端长为共，以尺价乘之，加端长，共得一十一贯三百五十三文。[2] 问：端长、尺价各几何？

答曰：端长四丈八尺，尺价八十五文。

术曰：立天元一为端长，如积求之。得一千六百六十四万六千四百为益实，七千二百七十三为从廉，一为益隅，立方开之，[3] 得端长。合问。

【注释】

[1] 设天元一 x 为端长，记尺价为 w，此即：$\frac{4080}{x}=w$。（郭）

[2] 此即：$(x+w)w+x=11353$。（郭）

[3] 开方式的现代形式为：$-x^3+7273x^2-16646400=0$。（陈）

price per *chi*, 54 cash.

Process. Let the element *tian* be the price of one *chi*. From the statement we have 2862 for the negative *shi*, 1 for the negative *fang*, and 1 for the positive *yu*, a quadratic expression [2] whose root is the required price of one chi.

[Notes]

[1] Let the element *tian* be the price of one *chi*. Use x for the *tian*, let w be the length of *pi*. That is, $(x - w)x - x = 1314$. (G)

[2] The expression in modern form is the equation: $x^2 - x - 2862 = 0$. (C)

3. The price of a *duan* embroidery is 4080 cash.[1] The price of *chi* plus the number of *chi* in the *duan*, multiplied by the price of a *chi*, plus the number of *chi* in the *duan*, is equal to 11353.[2] Find the length of the *duan* and the price of one *chi*.

Ans. Length, 4 *zhang* 8 *chi*;

price per *chi*, 85 cash.

Process. Let the element *tian* be the length of the *duan*. From the statement we have 16646400 for the negative *shi*, 7273 for the positive *lian*, and 1 for the negative *yu*, a cubic expression[3] whose root is the required length of the *duan*.

[Notes]

[1] Let the element *tian* be the length of the *duan*. Use x for the *tian*, let w be the

【今译】

今有锦1端，值钱4贯80文。只云尺价与端长相加，以尺价乘之，加端长，共得11贯353文。问：端长、尺价各为多少？

答：端长4丈8尺，尺价85文。

术：设天元一为端长，以如积方法求其解。得到−16646400为常数项，7273为二次项系数，−1为最高次项系数，开立方，便得到端长。符合所问。

4.

【原文】

今有锦一端、一匹，端长自乘，内减匹长；又匹长自乘，内减端长；二余相并，共得三千五百一十六尺。[1]只云端长多于匹长四分之一。[2]问：端、匹各长几何？

答曰：端长四丈八尺，匹长三丈六尺。

术曰：立天元一为端长，如积求之。得五万六千二百五十六为益实，二十八为益方，二十五为正隅，平方开之，[3]得端长。合问。

【注释】

[1] 设天元一x为端长，记匹长为w，此即：$(x^2-w)+(w^2-x)=3516$。（郭）

[2] 此即：$x-w=\frac{1}{4}x$。（郭）

[3] 开方式的现代形式为：$25x^2-28x-56256=0$。（陈）

【今译】

今有锦1端、1匹，端长自乘，减去匹长；又匹长自乘，减去端长；二者相加，共得3516尺。只云端长比匹长多端长的$\frac{1}{4}$。问：端长、匹长各为多少？

答：端长4丈8尺，匹长3丈6尺。

术：设天元一为端长，以如积方法求其解。得到−56256为常数项，−28为一次项系数，25为最高次项系数，开平方，便得到端长。符合所问。

price of one *chi*. That is, $\frac{4080}{x} = w$. (G)

[2] That is, $(x + w)w + x = 11353$. (G)

[3] The expression in modern form is the equation: $-x^3 + 7273x^2 - 16646400 = 0$. (C)

4. There are a *duan* and a *pi* of embroidery. The difference between the square of the *duan* and the length of the *pi* plus the difference between the square of the *pi* and the length of the *duan* is equal to 3516 *chi*.[1] It is said that the length of the *duan* exceeds the length of the *pi* by one-fourth of itself.[2] What is the length of each?

Ans. Length of the *duan*, 4 *zhang* 8 *chi*;

length of the *pi*, 3 *zhang* 6 *chi*.

Process. Let the element *tian* be the length of the *duan*. From the statement we have 56256 for the negative *shi*, 28 for the negative *fang*, and 25 for the positive *yu*, a quadratic expression[3] whose root is the required length of the *duan*.

[Notes]

[1] Let the element *tian* be the length of the *duan*. Use x for the *tian*, let w be the length of the *pi*. That is, $(x^2 - w) + (w^2 - x) = 3516$. (G)

[2] That is, $x - w = \frac{1}{4}x$. (G)

[3] The expression in modern form is the equation: $25x^2 - 28x - 56256 = 0$. (C)

5.

【原文】

今有绢一匹，直钱一贯六百六十六文。[1] 只云匹长如尺价五百四十四分之四百四十一。[2] 问：匹长、尺价各几何？

答曰：匹长三丈六尺四分尺之三，尺价四十五文三分文之一。

术曰：立天元一为匹长、尺价齐率[3]，如积求之。得一百四十四为益实，一为正隅，平方开之，[4] 得一十二为齐率，以除分母、子之数。合问。

【注释】

[1] 记 v 为匹长，w 为尺价，此即：$\frac{1666}{v}=w$。（郭）

[2] 此即：$v=\frac{441}{544}w$。（郭）

[3] 令 $\frac{x}{544}$ 为尺价，$\frac{441}{x}$ 为匹长，x 为匹长与尺价的齐率。（陈）

[4] 开方式的现代形式为：$x^2-144=0$。（陈）

【今译】

今有绢1匹，值钱1贯666文。只云匹长是尺价的 $\frac{441}{544}$。问：匹长、尺价各为多少？

答：匹长3丈 $6\frac{3}{4}$ 尺，尺价 $45\frac{1}{3}$ 文。

术：设天元一为匹长、尺价的齐率，以如积方法求其解。得到 -144 为常数项，1为最高次项系数，开平方，得到12为齐率。以齐率分别除上述的分子、分母，便得匹长、尺价。符合所问。

6.

【原文】

今有锦一匹，先卖了三尺，余卖得钱二贯九百七十五文。[1] 只云匹长不及尺价四十七文。[2] 问：匹长、尺价各几何？

答曰：匹长三丈八尺，尺价八十五文。

5. The price of a *pi* of silk is 1666 cash.[1] It is said that the length of the *pi* is $\frac{441}{544}$ of the price of one *chi*.[2] Find the length of *pi* and the price of one *chi*.

Ans. Length, 3 *zhang* $6\frac{3}{4}$ *chi*;

price per *chi*, $45\frac{1}{3}$ cash.

Process. Let the element *tian* be the *qi lu* [3] of the *pi* and also the price of the *chi*. From the statement we have 144 for the negative *shi* and 1 for the positive *yu*, a quadratic expression[4] whose root, 12, is the required *qi lu*. Dividing the numerator and the denominator of the given fraction by the *qi lu*, we have the required number.

[Notes]

[1] Let the length of *pi* be v, and the price of one *chi* w. That is, $\frac{1666}{v} = w$. (G)

[2] That is, $v = \frac{441}{544}w$. (G)

[3] Let $\frac{x}{544}$ be the price of one *chi*, $\frac{441}{x}$ be the length of the *pi*, and x be the *qi lu* of the length of the *pi* and also the price of one *chi*. (C)

[4] The expression in modern form is the equation: $x^2 - 144 = 0$. (C)

6. A person sold 3 *chi* from a piece of embroidery and afterward sold the rest for 2975 cash.[1] It is said that the whole length was less than the price of a *chi* by 47.[2] Find the length and one *chi*'s price.

Ans. Length, 3 *zhang* 8 *chi*;

术曰：立天元一为匹长，如积求之。得三千一百一十六为益实，四十四为从方，一为正隅，平方开之，[3]得匹长。　又：立天元一为尺价，如积求之。得二千九百七十五为益实，五十为益方，一为正隅，平方开之，[4]得尺价。合问。

【注释】

[1] 设天元一 x 为匹长，y 为尺价，此即：$y(x-3)=2975$。（郭）

[2] 此即：$y-x=47$。（郭）

[3] 开方式的现代形式为：$x^2+44x-3116=0$。（陈）

[4] 开方式的现代形式为：$y^2-50y-2975=0$。（陈）

【今译】

今有锦1匹，先卖了3尺，余下的卖了2贯975文。只云匹长比尺价少47文。问：匹长、尺价各为多少？

答：匹长3丈8尺，尺价85文。

术：设天元一为匹长，以如积方法求其解。得到 −3116 为常数项，44为一次项系数，1为最高次项系数，开平方，得到匹长。　又术：设天元一为尺价，以如积方法求其解。得到 −2975 为常数项，−50 为一次项系数，1为最高次项系数，开平方，便得到尺价。符合所问。

7.

【原文】

今有钱一百六十二贯五百六十文，买布不知匹数。[1]只云每匹牙钱五文，今无牙钱，准布二匹。[2]问：共布及匹价各几何？

答曰：二百五十六匹，匹价六百三十五文。

术曰：立天元一为共布，如积求之。得六万五千二十四为益实，二为

price per *chi*, 85 cash.

Process. Let the element *tian* be the length of the whole piece. From the statement we have 3116 for the negative *shi*, 44 for the positive *fang*, and 1 for the positive *yu*, a quadratic expression [3] whose root is the required length. Again let the element *tian* be the price of one *chi*. we have 2975 for the negative *shi*, 50 for the negative *fang*, and 1 for the positive *yu*, a quadratic expression [4] whose root is the required price.

[Notes]

[1] Let the element *tian* be the length of the *pi*. Use x for the *tian*. Let y be the price of one *chi*. That is, $y(x-3)=2975$. (G)

[2] That is, $y-x=47$. (G)

[3] The expression in modern form is the equation: $x^2+44x-3116=0$. (C)

[4] The expression in modern form is the equation: $y^2-50y-2975=0$. (C)

7. A man bought some piece of cloth for 162560 cash;[1] the commission fee on each piece was 5 cash. Since he had spent all his money for the cloth, he saved the commission fee by giving two pieces of cloth.[2] Find the number of pieces he bought and the price of one piece.

Ans. 256 pieces;

益方，一为从隅，平方开之，[3] 得共布之数。 又：立天元一为匹价，如积求之。得四十万六千四百为益实，五为从方，一为从隅，平方开之，[4] 得匹价。合问。

【注释】

[1] 设天元一 x 为共布匹数，y 为匹价，此即：$\frac{162560}{x}=y$。（郭）

[2] 牙钱是牙人收取的佣金。牙人是撮合买卖双方的人。此即：$\frac{5x}{635+5}=2$。（郭）

[3] 开方式的现代形式为：$x^2-2x-65024=0$。（陈）

[4] 开方式的现代形式为：$y^2+5y-406400=0$。（陈）

【今译】

今有钱162贯560文，买布不知匹数。只云每匹需牙钱5文，现在不给牙钱，需准折2匹布。问：总共的布数及匹价各为多少？

答：256匹，每匹价钱635文。

术：设天元一为总共的布数，以如积方法求其解。得到 −65024 为常数项，−2为一次项系数，1为最高次项系数，开平方，得到总共的布数。 又术：设天元一为匹价，以如积方法求其解。得到 −406400 为常数项，5为一次项系数，1为最高次项系数，开平方，便得到匹价。符合所问。

8.

【原文】

今有纱一匹，先截一尺，作牙钱，余，卖得钱一贯一百七十六文。[1] 只云：匹长、尺价皆以平方开之，二数相并，共得十二。[2] 问：匹长、尺价各几何？

答曰：匹长二丈五尺，尺价四十九文。

术曰：立天元一为匹长开方数，如积求之。得一千三百二十为益实，二十四为从方，一百四十三为从上廉，二十四为益下廉，一为正隅，三乘方开之，[3] 得五，为匹长开方数。合问。

price per piece, 635 cash.

Process. Let the element *tian* be the number of piece. From the statement we have 65024 for the negative *shi*, 2 for the negative *fang*, and 1 for the positive *yu*, a quadratic expression[3] whose root is the required number.

Again let the element *tian* be the price of one piece of cloth. From the statement we have 406400 for the negative *shi*, 5 for the positive *fang*, and 1 for the positive *yu*, a quadratic expression[4] whose root is the required number.

【Notes】

[1] Let the element *tian* be the number of pieces of the cloth. Use x for the *tian*. Let y be the price of one piece. That is, $\frac{162560}{x} = y$. (G)

[2] The *ya qian* is the commission fee which is received by *Ya ren*. The *Ya ren* acts as a go-between person to make the business succeed. That is, $\frac{5x}{635+5} = 2$. (G)

[3] The expression in modern form is the equation: $x^2 - 2x - 65024 = 0$. (C)

[4] The expression in modern form is the equation: $y^2 + 5y - 406400 = 0$. (C)

8. After cutting a *chi* from a *pi* of gauze to pay the commission fee, the rest is sold for 1176 cash.[1] It is said that the sum of the square roots of the length and of the price of one *chi* equals 12.[2] Find the length and one *chi*'s price.

Ans. Length, 2 *zhang* 5 *chi*;

price per *chi*, 49 cash.

Process. Let the element *tian* be the square root of the length. From the statement we have 1320 for the negative *shi*, 24 for the positive *fang*, 143

【注释】

[1] 记匹长为v，尺价为w，此即：$w(v-1)=1176$。(郭)

[2] 此即：$\sqrt{v}+\sqrt{w}=12$。(郭)

[3] 开方式的现代形式为：$x^4-24x^3+143x^2+24x-1320=0$。(陈)

【今译】

今有纱1匹，先截取1尺作牙钱，剩余的卖得钱1贯176文。只云：对匹长、尺价皆开平方，二者相加，共得12。问：匹长、尺价各为多少?

答：匹长2丈5尺，尺价49文。

术：设天元一为匹长的开方数，以如积方法求其解。得到−1320为常数项，24为一次项系数，143为二次项系数，−24为三次项系数，1为最高次项系数，开四次方，得到5，便是匹长开方数。符合所问。

9.

【原文】

今有绫、罗共三丈，各直钱八百九十六文。[1] 只云绫、罗各一尺，共直钱一百二十文。[2] 问：绫、罗尺、价各几何?

答曰：绫一丈四尺，尺价六十四文；罗一丈六尺，尺价五十六文。

术曰：立天元一为绫尺数，如积求之。得二百二十四为正实，三十为益方，一为正隅，平方开之，[3] 得绫尺数。　又：立天元一为绫尺价，如积求之。得三千五百八十四为益实，一百二十为从方，一为益隅，平方开之，[4] 得绫尺价。　又：立天元一为罗尺数，如积求之。得二百二十四为正实，三十为益方，一为正隅，平方开之，[5] 得罗尺数。

又：立天元一为罗尺价，如积求之。得三千五百八十四为正实，一百二十为益方，一为正隅，平方开之，[6] 得罗尺价。合问。

for the positive first *lian*, 24 for the negative last *lian*, and 1 for the positive *yu*, an expression [3] of the fourth degree whose root, 5, is the required square root of the length.

[Notes]

[1] Let the length of the *pi* be v, and the price of one *chi* be w. That is, $w(v-1) = 1176$. (G)

[2] That is, $\sqrt{v} + \sqrt{w} = 12$. (G)

[3] The expression in modern form is the equation: $x^4 - 24x^3 + 143x^2 + 24x - 1320 = 0$. (C)

9. There are three *zhang* of damask and gauze. The price of each piece is 896 cash.[1] It is said that if we take a *chi* of each the price will be 120 cash.[2] Find the price of one *chi* of each kind and the length of each piece.

Ans. Damask, length, 1 *zhang* 4 *chi*; price per *chi*, 64 cash;

gauze, length, 1 *zhang* 6 *chi*; price per *chi*, 56 cash.

Process. Let the element *tian* be the number of *chi* in the damask. From the statement we have 224 for the positive *shi*, 30 for the negative *fang*, and 1 for the positive *yu*, a quadratic expression [3] whose root is the required number. Again let the element *tian* be the price of one *chi* of damask. We have 3584 for the negative *shi*, 120 for the positive *fang*, and 1 for the negative *yu*, a quadratic expression[4] whose root is the price of one *chi* of

【注释】

[1] 设天元一x_1为绫尺数，y_1为绫尺价，x_2为罗尺数，y_2为罗尺价，此即：$x_1 + x_2 = 30$，$x_1y_1 = 896$，$x_2y_2 = 896$。（郭）

[2] 此即：$y_1 + y_2 = 120$。（郭）

[3] 开方式的现代形式为：$x^2_1 - 30x_1 + 224 = 0$。（陈）

[4] 开方式的现代形式为：$-y^2_1 + 120y_1 - 3584 = 0$。（陈）

[5] 开方式的现代形式为：$x^2_2 - 30x_2 + 224 = 0$。此开方式与求绫尺数者相同，这是一个二次方程有2个根的例子。（陈）

[6] 开方式的现代形式为：$y^2_2 - 120y_2 + 3584 = 0$。此开方式实际上与绫尺价者相同，亦是一个二次方程有2个根的例子。（陈）

【今译】

今有绫、罗共3丈，各值钱896文。只云绫、罗各1尺，共值钱120文。问：绫、罗的尺数和价钱各为多少？

答：绫1丈4尺，尺价64文；罗1丈6尺，尺价56文。

术：设天元一为绫尺数，以如积方法求其解。得到224为常数项，−30为一次项系数，1为最高次项系数，开平方，得到绫尺数。 又术：设天元一为绫尺价，以如积方法求其解。得到−3584为常数项，120为一次项系数，−1为最高次项系数，开平方，得到绫尺价。 又术：设天元一为罗尺数，以如积方法求其解。得到224为常数项，−30为一次项系数，1为最高次项系数，开平方，得到罗尺数。 又术：设天元一为罗尺价，以如积方法求其解。得到3584为常数项，−120为一次项系数，1为最高次项系数，开平方，便得到罗尺价。符合所问。

damask. Again let the element *tian* be the number of *chi* of gauze. From the statement we have 224 for the positive *shi*, 30 for the negative *fang*, and 1 for the positive *yu*, a quadratic expression[5] whose root is the number of *chi* of gauze. Again let the element *tian* be the price of one *chi* of gauze. We have 3584 for the positive *shi*, 120 for the negative *fang*, and 1 for the positive *yu*, a quadratic expression[6] whose root is the price of one *chi* of gauze.

【 Notes 】

[1] Let the element *tian* be the number of the damask. Use x_1 for the *tian*. Let y_1 be the price of one *chi* in the damask, x_2 the number of the gauze, y_2 the price of one *chi* in the gauze. That is, $x_1 + x_2 = 30$, $x_1y_1 = 896$, $x_2y_2 = 896$. (G)

[2] That is, $y_1 + y_2 = 120$. (G)

[3] The expression in modern form is the equation: $x_1^2 - 30x_1 + 224 = 0$. (C)

[4] The expression in modern form is the equation: $-y_1^2 + 120y_1 - 3584 = 0$. (C)

[5] The expression in modern form is the equation: $x_2^2 - 30x_2 + 224 = 0$. In fact, this equation is the same as the equation for the number of *chi* in the damask. It is an example of a quadratic equation with two roots. (C)

[6] The expression in modern form is the equation: $y_2^2 - 120y_2 + 3584 = 0$. In fact, this equation is the same as the equation for the price of one *chi* of damask. It is also an example of a quadratic equation with two roots. (C)

廪粟回求 六问

1.

【原文】

今有方仓一所，受粟五百七十六斛。[1] 只云仓阔不及仓长三尺，深如阔三分之二；[2] 斛法二尺五寸。后皆仿此。问：仓长、阔、深各几何？

答曰：长一丈五尺，阔一丈二尺，深八尺。

术曰：立天元一为仓长，如积求之。得二千一百六十为益实，九为从方，六为益廉，一为正隅，立方开之，[3] 得仓长。合问。

【注释】

[1] 记仓长为b，仓阔为a，仓深为c。斛法$2\frac{1}{2}$尺^3。此即：$abc \div 2\frac{1}{2} = 576$。（郭）

[2] 此即：$b - a = 3$，$c = \frac{2}{3}a$。（郭）

[3] 开方式的现代形式为：$x^3 - 6x^2 + 9x - 2160 = 0$。（陈）

【今译】

今有方仓一所，容纳粟576斛。只云仓阔比仓长少3尺，深是阔的$\frac{2}{3}$。斛法是$2\frac{1}{2}$寸。后皆仿此。问：方仓的长、阔、深各为多少？

答：长1丈5尺，阔1丈2尺，深8尺。

术：设天元一为仓长，以如积方法求其解。得到-2160为常数项，9为一次项系数，-6为二次项系数，1为最高次项系数，开立方，便得到符合问题的仓长。

2.

【原文】

今有圆囤，贮粟三百六十四斛五分斛之四。[1] 只云上周如下周太半，高如下周少半。[2] 问：周、高各几何？

Lin Su Hui Qiu (Problems on Store House for Grain)

6 Problems

1. The capacity of a *fang cang* is 576 *hu*.[1] It is said that the width of the *fang cang* is less than the length by 3 *chi* and the depth is two-thirds of the width;[2] 2 *chi* 5 *cun* equals one *hu*. Find the length, width, and depth of the *fang cang*.

Ans. Length, 15 *chi*; width, 12 *chi*; depth, 8 *chi*.

Process. Let the element *tian* be the length of the *fang cang*. From the statement we have 2160 for the negative *shi*, 9 for the positive *fang*, 6 for the negative *lian*, and 1 for the positive *yu*, a cubic expression [3] whose root is the required length.

[Notes]

[1] Let the length of the *fang cang* be b, the width a, and the depth c. $2\frac{1}{2}$ cubic *cun* equals one *hu*. That is, $abc \div 2\frac{1}{2} = 576$. (G)

[2] That is, $b - a = 3$, $c = \frac{2}{3}a$. (G)

[3] The expression in modern form is the equation: $x^3 - 6x^2 + 9x - 2160 = 0$. (C)

2. The capacity of a *yuan tun* is $364\frac{4}{5}$ *hu*[1]. It is said that the *shang zhou* is the great-half of the *xia zhou* and the height is the small-half of the *xia zhou* [2]. Find the *zhou* and the height of the *tun*.

答曰：上周二丈四尺，下周三丈六尺，高一丈二尺。

术曰：立天元一为上周，如积求之。得一万三千八百二十四为益实，一为正隅，立方开之，[3] 得上周。合问。

【注释】

[1] 圆囤即《九章算术》的圆亭，记其上周为 l_1，下周为 l_2，高为 h，《九章算术》给出其体积公式为 $V=\frac{1}{36}(l_1l_2+l_1^2+l_2^2)h$。《四元玉鉴》使用《九章算术》的公式。粟的斛法是 $2\frac{1}{2}$ 尺^3。此条题设即：$\frac{1}{36}(l_1l_2+l_1^2+l_2^2)h\div2\frac{1}{2}=364\frac{4}{5}$。（郭）

[2] 此即：$l_1=\frac{2}{3}l_2$，$h=\frac{1}{3}l_2$。（郭）

[3] 开方式的现代形式为：$x^3-13824=0$。（陈）

【今译】

今有圆囤，贮存粟 $364\frac{4}{5}$ 斛。只云上周是下周的 $\frac{2}{3}$，高是下周的 $\frac{1}{3}$。问：上周、下周、高各为多少？

答：上周2丈4尺，下周3丈6尺，高1丈2尺。

术：设天元一为上周，以如积方法求其解。得到 −13824 为常数项，1为最高次项系数，开立方，便得到上周。符合所问。

3.

【原文】

今有圆囤，高一丈二尺，周四丈八尺。[1] 盛粟满中而适尽。只云今已运出三百八十四斛[2]。问：余粟残深几何？

答曰：残深七尺。

术曰：立天元一为残深，如积求之。得一千八为益实，一百四十四为从方，开无隅平方而一，[3] 得残深。合问。

Ans. *Shang zhou*, 2 *zhang* 4 *chi*;

xia zhou, 3 *zhang* 6 *chi*;

height, 1 *zhang* 2 *chi*.

Process. Let the element *tian* be the *shang zhou*. From the statement we have 13824 for the negative *shi* and 1 for the positive *yu*, a cubic expression [3] whose root is the required *shang zhou*.

[Notes]

[1] A *yuan tun* is a bin in the form of a frustum of a cone, the same as the *yuan ting* in *The Nine Chapters of Mathematical Procedures*.Let its *shang zhou* be l_1, *xia zhou* l_2, height h. The formula of its volume given by *The Nine Chapters of Mathematical Procedures* is as follows: $V = \frac{1}{36}(l_1l_2 + l^2_1 + l^2_2)h$. The *Jade Mirror of the Four Unknowns* used the formula in *The Nine Chapters of Mathematical Procedures*. The volume of one hu is $2\frac{1}{2}$ chi^3. This statement means: $\frac{1}{36}(l_1l_2 + l^2_1 + l^2_2)h \div 2\frac{1}{2} = 364\frac{4}{5}$. (G)

[2] That is, $l_1 = \frac{2}{3}l_2$, $h = \frac{1}{3}l_2$. (G)

[3] The expression in modern form is the equation: $x^3 - 13824 = 0$. (C)

3. The height of a *yuan tun* which is filled with grain is 12 *chi* and its circumference 48 *chi*.[1] After taking out 384 *hu* [2], what will be the depth of the remaining grain?

Ans. 7 *chi*.

Process. Let the element *tian* be the depth of the remaining grain. From the statement we have 1008 for the negative *shi* and 144 for the positive *fang*, an expression[3] without any *yu* whose root is the required depth.

【注释】

[1] 此圆囤即圆柱体，记其周为l，高为h，《九章算术》给出其体积公式为：$V=\frac{1}{12}l^2h$。《四元玉鉴》使用《九章算术》的公式。（郭）

[2] 设天元一x为残深，运出的粟在圆囤中的高为$12-x$，因此$\frac{1}{12}l^2(12-x)=2\frac{1}{2}\times 384$。（郭）

[3] 开方式的现代形式为：$144x-1008=0$。朱世杰称为开无隅平方。（陈）

【今译】

今有圆囤，高1丈2尺，周4丈8尺。盛满了粟，恰好到顶。只云现在运出去了384斛。问：剩余的粟的残深为多少？

答：残深为7尺。

术：设天元一为残深，以如积方法求其解。得到-1008为常数项，144为一次项系数，开无隅平方，以方除隅，便得到残深。符合所问。

4.

【原文】

今有方仓、圆囤各一所，贮粟三千三百一十二斛。[1] 只云仓广少于仓长四尺，多于仓深二尺，又多囤径二分之一，却与囤高等。[2] 问：仓、囤高、深、长、广各几何？

答曰：仓广一丈八尺，长二丈二尺，深一丈六尺；

囤径一丈二尺，高一丈八尺，周三丈六尺。

术曰：立天元一为仓广，如积求之。得一万二千四百二十为益实，一十二为益方，三为从廉，二为正隅，立方开之，[3] 得仓广。合问。

【注释】

[1] 此圆囤亦为圆柱体。记方仓的广、长、深分别为a，b，c；圆囤的周、高、径分别为l，h，d，此即：$\left(abc+\frac{1}{12}l^2h\right)\div 2\frac{1}{2}=3312$。（郭）

[Notes]

[1] The *yuan tun* is a bin in the form of a cylinder. Let its circumference be l, and its height h. *The Nine Chapters of Mathematical Procedures* has provided its formula of volume, that is, $V = \frac{1}{12} l^2 h$. The *Jade Mirror of the Four Unknowns* used the formula in *The Nine Chapters of Mathematical Procedures*. (G)

[2] Let the element *tian* be the depth of the remaining grain. Use x for the *tian*. The height of the grain taken out of the *yuan tun* is $12 - x$. Therefore, $\frac{1}{12} l^2 (12 - x) = 2\frac{1}{2} \times 384$. (G)

[3] The expression in modern form is the equation: $144x - 1008 = 0$. Zhu Shijie called it extracting an equation without any *yu*. (C)

4. A *fang cang* and a *yuan tun* contain 3312 *hu* of grain.[1] The width of the *cang* is less by 4 *chi* than the length, but exceeds its depth by two *chi*; it exceeds also the diameter of the *tun* by one-half and equals the height of the *tun*. [2] Find the heights, lengths, and widths of each.

Ans. *Cang*, width, 18 *chi*; length 22 *chi*; height, 16 *chi*.

Tun, diameter, 12 *chi*; height, 18 *chi*; circumference, 36 *chi*.

Process. Let the element *tian* be the width of the *cang*. From the statement we have 12420 for the negative *shi*, 12 for the negative *fang*, 3 for the positive *lian*, and 2 for the positive *yu*, a cubic expression[3] whose root is the required width.

[Notes]

[1] The *yuan tun* is also a bin in the form of a cylinder. Let *fang cang*'s width,

［2］此即：$b-a=4$，$a-c=2$，$a-d=\frac{1}{2}d$，$a=h$。（郭）

［3］开方式的现代形式为：$2x^3+3x^2-12x-12420=0$。（陈）

【今译】

今有方仓、圆囤各一所，贮存粟3312斛。只云仓广比仓长少4尺，比仓深多2尺，比囤径多囤径的$\frac{1}{2}$，而与囤高相等。问：方仓的广、长、深，圆囤的直径、高、周各为多少？

答：方仓的广1丈8尺，长2丈2尺，深1丈6尺；

圆囤的直径1丈2尺，高1丈8尺，周3丈6尺。

术：设天元一为仓广，以如积方法求其解。得到-12420为常数项，-12为一次项系数，3为二次项系数，2为最高次项系数，开立方，便得到仓广。符合所问。

5.

【原文】

今有方仓四、圆囤五，受粟四千七百六十八斛。[1] 只云仓方取中半，自乘，减七尺，余与囤高等；又囤径取中半，自乘，加三尺，却与仓深同；仓方多于囤径二尺。[2] 问：仓、囤高、深、方、径各几何？

答曰：仓方一丈，深一丈九尺；

囤径八尺，高一丈八尺。

术曰：立天元一为仓半方面，如积求之。得一万二千二十五为益实，二百一十为从方，二十六为益上廉，六十二为益下廉，三十一为从隅，三乘方开之，[3] 得半方仓面。合问。

【注释】

［1］此方仓为方柱体，圆囤亦为圆柱体。记方仓的方、深分别为a，b；圆囤的高、径分别为h，d，此即：$(4a^2b+5\times\frac{3}{4}d^2h)\div2\frac{1}{2}=4768$。（郭）

length, and height be a, b, and c. Let *yuan tun*'s circumference, height, and diameter be l, h, and d. That is, $\left(abc + \frac{1}{12}l^2h\right) \div 2\frac{1}{2} = 3312$. (G)

[2] That is, $b - a = 4, a - c = 2, a - d = \frac{1}{2}d, a = h$. (G)

[3] The expression in modern form is the equation: $2x^3 + 3x^2 - 12x - 12420 = 0$. (C)

5. There are four *fang cang* and five *yuan tun* containing 4768 *hu* of grain.[1] It is said that the square of one-half of the side of the *cang* diminished by 7 *chi* is equal to the height of the *tun*; the square of one-half of the diameter of the *tun* increased by 3 *chi* is equal to the depth of the *cang*; and a side of the base of the *cang* exceeds the diameter of the *tun* by 2 *chi*.[2] Find the height, the diameter, and a side of each.

Ans. *Cang*, side, 1 *zhang*; depth, 1 *zhang* 9 *chi*.

Tun, diameter, 8 *chi*; height, 1 *zhang* 8 *chi*.

Process. Let the element *tian* be one-half of a side of the base of the *cang*. From the statement we have 12025 for the negative *shi*, 210 for the positive *fang*, 26 for the negative first *lian*, 62 for the negative last *lian*, and 31 for the positive *yu*, an expression[3] of the fourth degree whose root is the

[2] 此即：$(\frac{1}{2}a)^2-7=h$；$(\frac{1}{2}d)^2+3=b$；$a-d=2$。（郭）

[3] 开方式的现代形式为：$31x^4-62x^3-26x^2+210x-12025=0$。（陈）

【今译】

今有4所方仓、5所圆囤，贮存粟4768斛。只云：取方仓的方边的$\frac{1}{2}$，自乘，减7尺，与圆囤的高相等；圆囤直径的$\frac{1}{2}$，自乘，加上3尺，与方仓的深相等；方仓的方边比圆囤直径多2尺。问：方仓的方、深，圆囤的直径、高各为多少？

答：方仓的方1丈，深1丈9尺；

圆囤的直径8尺，高1丈8尺。

术：设天元一为方仓的方边的$\frac{1}{2}$，以如积方法求其解。得到-12025为常数项，210为一次项系数，-26为二次项系数，-62为三次项系数，31为最高次项系数，开四次方，便得到方仓的方边的$\frac{1}{2}$。符合所问。

6.

【原文】

今有粟一千九十六斛八斗，用仓、囤各一贮之。不尽者平地堆之。[1] 只云仓长多于仓深七尺，不及囤周二丈；仓深却多平地粟高三尺；仓阔如仓长二分之一；圆囤周、高和得四十八尺；其平地粟高自乘，加入粟高，与粟周等。[2] 问：三事各得几何？

答曰：仓长一丈六尺，阔八尺，深九尺；

囤周三丈六尺，高一丈二尺；

粟周四丈二尺，高六尺。

术曰：立天元一为仓深，如积求之。得五万二千八百九十三为益实，二千三百一十三为从方，一十八为益上廉，八十二为从二廉，一十三为益下廉，一为正隅，四乘方开之，[3] 得仓深。 又立天元一为仓长，如积求之。得一十四万六千一百一十二为益实，四万四千四百六十为

required one-half of a side of the base of the *cang*.

【Notes】

[1] The *fang cang* is a bin whose base is a square, the *yuan tun* is also a bin in the form of a cylinder. Let *fang cang*'s side be a, and its depth b. Let *yuan tun*'s height h, and its diameter d. That is, $(4a^2b + 5 \times \frac{3}{4}d^2h) \div 2\frac{1}{2} = 4768$. (G)

[2] It must be a mistake. Now I revise. That is, $(\frac{1}{2}a)^2 - 7 = h$; $(\frac{1}{2}d)^2 + 3 = b$; $a - d = 2$. (G)

[3] The expression in modern form is the equation: $31x^4 - 62x^3 - 26x^2 + 210x - 12025 = 0$. (C)

6. There are 1096 *hu* 8 *dou* of grain. One *cang* and one *tun* are filled from it and the rest is heaped on the floor. [1] It is said that the length of the *cang* exceeds its depth by 7 *chi*, but is less than the circumference of the *tun* by 20 *chi*. The depth of the *cang* exceeds the height of the heap by 3 *chi*; the width of the *cang* is one-half of its length; the sum of the circumference and the height of the *tun* is 48 *chi*; and the sum of the square of the height of the heap and the height of the heap is equal to the circumference of its base.[2] Find the dimensions of the three.

Ans. *Cang*, length, 1 *zhang* 6 *chi*; width, 8 *chi*; depth, 9 *chi*.

Tun, circumference, 3 *zhang* 6 *chi*; height, 1 *zhang* 2 *chi*.

Heap, circumference of the base, 4 *zhang* 2 *chi*; height, 6 *chi*.

从方，八千九百九十二为益上廉，九百三十六为从二廉，四十八为益下廉，一为正隅，四乘方开之，[4] 得仓长。 又立天元一为仓阔，如积求之。得一万八千二百六十四为益实，一万一千一百一十五为从方，四千四百九十六为益上廉，九百三十六为从二廉，九十六为益下廉，四为从隅，四乘方开之，[5] 得仓阔。 又立天元一为囤周，如积求之。得二千三百万一百一十二为益实，三百八十六万三千三百四十为从方，二十六万三百五十二为益上廉，八千七百七十六为从二廉，一百四十八为益下廉，一为正隅，四乘方开之，[6] 得囤周。 又立天元一为囤高，如积求之。得二百三十万二千九百九十二为正实，六十万八百七十六为益方，六万三千三百六十为从上廉，三千四百为益二廉，九十二为从下廉，一为益隅，四乘方开之，[7] 得囤高。 又立天元一为粟高，如积求之。得四万四千七百一十二为益实，三千四百二十为从方，二百八十八为从上廉，一十六为从二廉，二为从下廉，一为正隅，四乘方开之，[8] 得粟高。合问。

【注释】

[1] 此方仓为长方体，圆囤亦为圆柱体，平地堆粟为圆锥。记方仓的阔、长、深分别为 a, b, c; 圆囤的周、高分别为 l_1, h_1; 平地堆粟的周、高分别为 l_2, h_2,《九章算术》商功章委粟术给出其体积公式 $V_3 = \frac{1}{36}l^2_2h_2$; 8 斗为 $\frac{4}{5}$ 斛。此即：$(abc + \frac{1}{12}l_1^2h_1 + \frac{1}{36}l^2_2h_2) \div 2\frac{1}{2} = 1096\frac{4}{5}$。（郭）

[2] 此即：$b - c = 7$，$l_1 - b = 20$，$c - h_2 = 3$，$a = \frac{1}{2}b$，$h + l_1 = 48$，$h^2_2 + h_2 = l_2$。（郭）

[3] 开方式的现代形式为：$x^5 - 13x^4 + 82x^3 - 18x^2 + 2313x - 52893 = 0$。（陈）

[4] 开方式的现代形式为：$x^5 - 48x^4 + 936x^3 - 8992x^2 + 44460x - 146112 = 0$。（陈）

[5] 开方式的现代形式为：$4x^5 - 96x^4 + 936x^3 - 4496x^2 + 11115x - 18264 = 0$。（陈）

Process. Let the element *tian* be the depth of the *cang*. From the statement we have 52893 for the negative *shi*, 2313 for the positive *fang*, 18 for the negative first *lian*, 82 for the positive second *lian*, 13 for the negative last *lian*, and 1 for the positive *yu*, an expression [3] of the fifth degree whose root is the required depth of the *cang*. Again let the element *tian* be the length of the *cang*. From the statement we have 146112 for the negative *shi*, 44460 for the positive *fang*, 8992 for the negative first *lian*, 936 for the positive second *lian*, 48 for the negative last *lian*, and 1 for the positive *yu*, an expression [4] of the fifth degree whose root is the length of the *cang*. Again let the element *tian* be the width of the *cang*. We have 18264 for the negative *shi*, 11115 for the positive *fang*, 4496 for the negative first *lian*, 936 for the positive second *lian*, 96 for the negative last *lian*, and 4 for the positive *yu*, an expression [5] of the fifth degree whose root is the width of the *cang*. Again let the element *tian* be the circumference of the *tun*. From the statement we have 23000112 for the negative *shi*, 3863340 for the positive *fang*, 260352 for the negative first *lian*, 8776 for the positive second *lian*, 148 for the negative last *lian*, and 1 for the positive *yu*, an expression [6] of the fifth degree whose root is the circumference of the *tun*. Again let the element *tian* be the height of the *tun*. From the statement we have 2302992 for the positive *shi*, 600876 for the negative *fang*, 63360 for the positive first *lian*, 3400 for the negative second *lian*, 92 for the positive last *lian*, and 1 for the negative *yu*, an expression [7] of the fifth degree whose root is the height of the *tun*. Again let the element *tian* be the height of the heap of grain. From the statement we have 44712 for the negative *shi*, 3420 for the positive *fang*, 288 for the positive first *lian*, 16 for the positive second *lian*, 2 for the positive last *lian*, and 1 for the positive *yu*, an expression [8] of the fifth degree whose root is the height of the heap.

[6] 开方式的现代形式为：$x^5-148x^4+8776x^3-260352x^2+3863340x-23000112=0$。(陈)

[7] 开方式的现代形式为：$-x^5+92x^4-3400x^3+63360x^2-600876x+2302992=0$。(陈)

[8] 开方式的现代形式为：$x^5+2x^4+16x^3+288x^2+3420x-44712=0$。(陈)

【今译】

今有粟1096斛8斗，用方仓、圆囤各一所贮存，贮之不尽，堆放在平地上成为一个圆锥形。只云：方仓的长比深多7尺，比圆囤的周少2丈；方仓的深比圆锥的高多3尺；方仓的阔是长的$\frac{1}{2}$；圆囤的周、高之和为48尺；其圆锥的高加它的自乘，等于它的周。问：方仓的长、阔、深，圆囤、圆锥的周、高各为多少？

答：方仓的长1丈6尺，阔8尺，深9尺；

圆囤的周3丈6尺，高1丈2尺；

圆锥的周4丈2尺，高6尺。

术：设天元一为方仓的深，以如积方法求其解。得到-52893为常数项，2313为一次项系数，-18为二次项系数，82为三次项系数，-13为四次项系数，1为最高次项系数，开五次方，便得到符合问题的方仓的深。又术：设天元一为方仓的长，以如积方法求其解。得到-146112为常数项，44460为一次项系数，-8992为二次项系数，936为三次项系数，-48为四次项系数，1为最高次项系数，开五次方，便得到符合问题的方仓的长。又术：设天元一为方仓的阔，以如积方法求其解。得到-18264为常数项，11115为一次项系数，-4496为二次项系数，936为三次项系数，-96为四次项系数，4为最高次项系数，开五次方，便得到符合问题的方仓的阔。又术：设天元一为圆囤的周，以如积方法求其解。得到-23000112为常数项，3863340为一次项系数，-260352为二次项系数，8776为三次项系数，-148为四次项系数，1为最高次项系数，开五次方，便得到符合问题的圆囤的周。又术：设天元一为圆囤的高，以如积方法求其解。得到2302992为常数项，-600876为一次项系数，63360为二次项系数，-3400为三次项系数，92为四次项系数，-1为最高次项系数，开五次方，便得到符合问题的圆囤的高。又术：设天元一为圆锥的高，以如积方法求其解。得到-44712为常数项，3420为一次项系数，288为二次项系数，16为三次项系数，2为四次项系数，1为最高次项系数，开五次方，便得到圆锥的高。符合所问。

【 Notes 】

[1] The *fang cang* is a cuboid, the *yuan tun* is also a bin in the form of a cylinder, and the heaped grain on the floor is in the form of a cone. Let the *fang cang*'s width be a, length b, and depth c. Let the *yuan tun*'s circumference be l_1, and height h_1. Let the heaped grain's circumference be l_2, and height h_2. The volume's formula was given by the *wei su* method in the chapter *shang gong* (consultations on engineering works) in *The Nine Chapters of Mathematical Procedures*. It is as follows: $V = \frac{1}{36} l_2^2 h_2$. 8 *dou* is $\frac{4}{5}$ *hu*. That is, $(abc + \frac{1}{12} l_1^2 h_1 + \frac{1}{36} l_2^2 h_2) \div 2\frac{1}{2} = 1096\frac{4}{5}$. (G)

[2] That is, $b - c = 7, l_1 - b = 20, c - h_2 = 3, a = \frac{1}{2} b, h + l_1 = 48, h_2^2 + h_2 = l_2$. (G)

[3] The expression in modern form is the equation: $x^5 - 13x^4 + 82x^3 - 18x^2 + 2313x - 52893 = 0$. (C)

[4] The expression in modern form is the equation: $x^5 - 48x^4 + 936x^3 - 8992x^2 + 44460x - 146112 = 0$. (C)

[5] The expression in modern form is the equation: $4x^5 - 96x^4 + 936x^3 - 4496x^2 + 11115x - 18264 = 0$. (C)

[6] The expression in modern form is the equation: $x^5 - 148x^4 + 8776x^3 - 260352x^2 + 3863340x - 23000112 = 0$. (C)

[7] The expression in modern form is the equation: $-x^5 + 92x^4 - 3400x^3 + 63360x^2 - 600876x + 2302992 = 0$. (C)

[8] The expression in modern form is the equation: $x^5 + 2x^4 + 16x^3 + 288x^2 + 3420x - 44712 = 0$. (C)

商功修筑 七问

1.

【原文】

今有积筑圆城一座，计积四百八十八万五千三百四十四尺。[1] 只云：下内、外周差一百八尺，上内、外周差四丈二尺，上、下外周差六十尺，上、下内周差六尺，下广少如高六尺，却多上广一丈一尺，高不及上内周一万六千二百二十四尺。[2] 令侵城撅壕取土筑城，定壕广三丈。问：内、外周、高及上、下广，并壕深各得几何？

答曰：下外周九里三十步，内周九里八步二尺，广一丈八尺；

上外周九里一十八步，内周九里九步三尺，广七尺；

高二丈四尺；

深一丈三尺二寸一万二百七十五分寸之七十四。[3]

术曰：立天元一为城高，如积求之。得四百八十八万五千三百四十四为益实，一十八万六千九百四十八为益方，一万六千二百四十七为从廉，一为正隅，立方开之，[4] 得高二丈四尺。余依加减求之。 求壕深术曰：四因城积，三除为实。又城下外周并入六个壕广，及城外周折半，以壕广乘之，为法。实如法而一，即壕深。合问。[5]

【注释】

[1] 此圆城的形状与《九章算术》商功章的曲池相同，其术如刍童。记刍童的上、下广，上、下长，高分别为 a_1, a_2, b_1, b_2, h,《九章算术》给出的体积公式为:

$V=\frac{1}{6}[(2b_1+b_2)a_1+(2b_2+b_1)a_2]h$,

记曲池或圆城的上内、外周，下内、外周分别为 l_1, L_1, l_2, L_2,《九章算术》在刍童公式中令 $b_1=\frac{1}{2}(l_1+L_1)$, $b_2=\frac{1}{2}(l_2+L_2)$, 以求曲池（或圆城）的体积。此条题设即：$V=\frac{1}{6}\{[2\times\frac{1}{2}(l_1+L_1)+\frac{1}{2}(l_2+L_2)]a_1+[2\times\frac{1}{2}(l_2+L_2)+$

Shang Gong Xiu Zhu (Problems on Labor)

7 Problems

1. The volume of the wall of a round city is 4885344 *chi*.[1] It is said that the difference between the interior and exterior circumferences at the bottom of the wall is 108 *chi*, the difference between the circumferences at the top is 42 *chi*, the difference between the exterior circumferences of the top and the bottom is 60 *chi*, the interior circumferences differ from each other by 6 *chi*, the width at the bottom is less than the height by 6 *chi* but exceeds the width at the top by 11 *chi*, and the height is less than the interior circumference at the top by 16224 *chi*.[2] If a ditch whose width is 30 *chi* is dug close to the bottom of the wall and the earth is used for the building of the wall, what should be the depth of the ditch? What are the exterior and interior circumferences of both the top and the bottom, and the width and the height of the wall?

Ans. Bottom of the wall:

exterior circumference, 9 *li*, 30 *bu*;

interior circumference, 9 *li* 8 *bu* 2 *chi*;

width, 1 *zhang* 8 *chi*.

Top of the wall:

exterior circumference, 9 *li*, 18 *bu*;

interior circumference, 9 *li* 9 *bu* 3 *chi*;

width, 7 *chi*.

Height of the wall:

2 *zhang* 4 *chi*.

Depth of the ditch:

1 *zhang* 3 *chi* $2\frac{74}{10275}$ *cun*. [3]

Process. Let the element *tian* be the height of the city wall. From the statement we have 4885344 for the negative *shi*, 186948 for the negative *fang*, 16247 for the positive *lian*, and 1 for the positive *yu*, a cubic expression [4]

$\frac{1}{2}(l_1+L_1)]a_2\}h=4885344$。（郭）

［2］此即：$L_2-l_2=108$，$L_1-l_1=42$，$L_2-L_1=60$，$l_2-l_1=6$，$h-a_2=6$，$a_2-a_1=11$，$l_1-h=16224$。（郭）

［3］5尺＝1步，2步＝1丈，180丈＝360步＝1里。（陈）

［4］开方式的现代形式为：$x^3+16247x^2-186948x-4885344=0$。（陈）

［5］此壕沟的平截面为一圆环，壕沟的广即其径，记为a_3；其内周即圆城的下外周L_2，其外周记为L_3，取圆周率3，则$L_3=(\frac{L_2}{\pi}+2a_3)\pi=(\frac{L_2}{3}+2a_3)\times3=L_2+6a_3$，根据《九章算术》环田术，其面积为$\frac{1}{2}[L_2+(L_2+6a_3)]a_3$。记壕沟的深为$h_3$，则其体积为$\frac{1}{2}[L_2+(L_2+6a_3)]a_3h_3$。根据《九章算术》商功章：穿地4，为坚3，圆城的体积为$V=\frac{3}{4}\times\frac{1}{2}[L_2+(L_2+6a_3)]a_3h_3$，故$\frac{4V}{3}\div\frac{1}{2}[L_2+(L_2+6a_3)]a_3$即是壕深。（郭）

【今译】

今有土积修筑一座圆城，其体积为4885344尺。只云：下内、外周相差108尺，上内、外周相差4丈2尺，上、下外周相差60尺，上、下内周相差6尺，下广比高少6尺，却比上广多1丈1尺，高与上内周相差16224尺。今紧挨城墙挖掘壕沟，取土修筑圆城，确定壕沟广3丈。问：圆城的内、外周、高，上、下广，及壕沟的深各为多少？

答：圆城的下外周9里30步，内周9里8步2尺，广1丈8尺；

上外周9里18步，内周9里9步3尺，广7尺；

高2丈4尺。

壕沟的深1丈3尺$2\frac{74}{10275}$寸。

术：设天元一为城高，以如积方法求其解。得到-4885344为常数项，-186948为一次项系数，16247为二次项系数，1为最高次项系数，开立方，得到高2丈4尺。其他诸事由与城高的关系通过加减法求出。

whose root is 24, the required height of the wall. The other dimensions can be obtained from the addition and subtraction method. The method for finding the depth of the ditch is as follows: Multiply the volume of the wall by 4, then divide the product by 3, the result thus obtained is dividend. Find the sum of the exterior circumference at the bottom of the wall and 6 times the width of the ditch and this circumference, divide by 2, then multiply the quotient by the width of the ditch for the divisor. After dividing the dividend by the divisor we have the required depth. [5]

【Notes】

[1] The round city's form is the same as *qu chi* in the chapter *shang gong* of *The Nine Chapters of Mathematical Procedures*.Its method is like *chu tong*. Let *chu tong*'s upper width be a_1, lower width a_2, upper length b_1, lower length b_2, height h. Its volume formula given by *The Nine Chapters of Mathematical Procedures* is as follows:

$$V=\frac{1}{6}[(2b_1+b_2)a_1+(2b_2+b_1)a_2]h.$$

Let *qu chi* or round city's interior circumference at the top be l_1, exterior circumference at the top L_1, interior circumference at the bottom l_2, exterior circumference at the bottom top L_2. *The Nine Chapters of Mathematical Procedures* used $b_1=\frac{1}{2}(l_1+L_1)$, $b_2=\frac{1}{2}(l_2+L_2)$ in the formula of *chu tong* for the volume of *qu chi* (or round city). According to the statement, that is, $V=\frac{1}{6}\{[2\times\frac{1}{2}(l_1+L_1)+\frac{1}{2}(l_2+L_2)]a_1+[2\times\frac{1}{2}(l_2+L_2)+\frac{1}{2}(l_1+L_1)]a_2\}h=4885344$. (G)

[2] That is, $L_2-l_2=108$, $L_1-l_1=42$, $L_2-L_1=60$, $l_2-l_1=6$, $h-a_2=6$, $a_2-a_1=11$, $l_1-h=16224$. (G)

[3] 5 *chi* = 1 *bu*, 2 *bu* = 1 *zhang*, 180 *zhang* = 360 *bu* = 1 *li*. (C)

[4] The expression in modern form is the equation: $x^3+16247x^2-186948x-$

求壕沟深之术：以4乘圆城的体积，以3除之，作为实。又以圆城的下外周与壕沟广的6倍和下外周相加，除以2，以壕沟的广乘之，作为法。实除以法，便得到壕沟深。符合所问。

2.

【原文】

今有筑方城一座，计积四千五百四十一万七千六百尺。[1]只云：下面外方减十步，余，开方除之，并入下广，共得六十五步。又开方数少如上面外方三千五百四十六步。上面内、外方差四步。上面外方多如下面内方六步。上、下广差三步。上广不及高五步一尺。[2]令侵城四角，周回撅圈池取土筑城，及烧砖包城。令池上广三丈五尺，下广三丈。[3]计料：内、外城头合用条砖二千四百万个。其砖每个长一尺，阔五寸，厚二寸半。[4]每人日常役二十四尺，每人日烧砖及包讫城砖三十个。今差夫五万人，一齐兴功。问：上、下内、外方、广及高，并兴功毕日池深各几何？

答曰：下外方一十里一十步，内方一十里，广二丈五尺；

上外方一十里六步，内方一十里二步，广一丈；

4885344 = 0. (C)

[5] The horizontal section of the ditch is in the form of a ring. The ditch's width is the diameter. Use a_3 for it. The interior circumference is the exterior circumferences at the bottom of the round city, L_2. Use L_3 for the exterior circumference. The value of π is 3. Then, $L_3 = \left(\frac{L_2}{\pi} + 2a_3\right)\pi = \left(\frac{L_2}{3} + 2a_3\right) \times 3 = L_2 + 6a_3$. According to the *huan tian* method in *The Nine Chapters of Mathematical Procedures*, its area is $\frac{1}{2}[L_2 + (L_2 + 6a_3)]a_3$. Let the depth of the ditch be h_3. Its volume is $\frac{1}{2}[L_2 + (L_2 + 6a_3)]a_3h_3$. According to the chapter *shang gong* in *The Nine Chapters of Mathematical Procedures*, the volume of the round city is $V = \frac{3}{4} \times \frac{1}{2}[L_2 + (L_2 + 6a_3)]a_3h_3$. Therefore, $\frac{4V}{3} \div \frac{1}{2}[L_2 + (L_2 + 6a_3)]a_3$ is the depth of the ditch. (G)

2. The wall of a square city contains 45417600 *chi*.[1] It is said that the square root of an exterior side at the bottom diminished by 10 *bu* plus the width of the wall at the bottom is equal to 65 *bu*; the root is less than a side at the top of the wall 3546 *bu*; the difference between the interior and exterior sides at the top of the wall is 4 *bu*; the exterior side at the top exceeds the interior side at the bottom by 6 *bu*; the width at the top and the width at the bottom differ from each by 3 *bu*; and the width at the top is less than the height of the wall by 5 *bu* 1 *chi*.[2] A circular ditch is dug to circumscribe the wall in order to obtain earth for filling it and for making the bricks to build it. The width of the ditch at the top is 35 *chi* and at the bottom 30 *chi*. [3] 24000000 bricks are needed and the size of a brick is to be 1 *chi* in length, 5 *cun* in width, and 2.5

高三丈六尺；

池深二丈五尺五寸二千四百六十七分寸之四百八十三；

兴功五十一日一千一百二十五分日之四百七十三。

术曰：立天元一为下广，如积求之。得四十一万二千三百四十八为益实，一万一千一百四十八步六分为从方，一万四千八百九十八步二分为从上廉，一百三十一步八分为益下廉，[5]一为正隅，三乘方开之，[6]得下广。余依加减求之。 求池深术曰：列积，四之，三而一，于上。又一砖之积乘合用砖数，四之，五而一，加上，为实。[7] 又城外方身加[8]外四，三之，加六个池上广，为池外周。又池内周加六个池下阔，为池底外周。并而半之，为池底停周。又并池内、外周而半之，为池上停周。倍之，加底停周，以上广乘之，于上。又，倍底停周，加上停周，以下广乘之，并上，如六而一。所得为法。除实，即池深。[9]

求兴功毕日术曰：置城积，并入合用砖数，以二人乘之，为实。并人日常积及人日烧用砖数，以共差夫乘之，得数，为法。实如法而一。[10]合问。

【注释】

[1]记方城的上内、外方、广，下内、外方、广，高分别为l_1，L_1，a_1，l_2，L_2，a_2，h，其一侧的求积方法与曲池相同。整个方城的体积是一侧的4倍。此条题设即：

$$V=4\times\frac{1}{6}\{[2\times\frac{1}{2}(l_1+L_1)+\frac{1}{2}(l_2+L_2)]a_1+[2\times\frac{1}{2}(l_2+L_2)+\frac{1}{2}(l_1+L_1)]a_2\}h=45417600。$$（郭）

[2]以步为单位，此即$\sqrt{L_2-10}+a_2=65$，$L_1-\sqrt{L_2-10}=3546$，$L_1-l_1=4$，$L_1-l_2=6$，$a_2-a_1=3$，$h-a_1=5\frac{1}{5}$。（郭）

[3]此即于方城外接一圆环形曲池，实际上是取挖环池的土筑城。其上广$b_1=35$，下广$b_2=30$。环池的求积方法与《九章算术》的曲池术相同。（郭）

cun in thickness.[4] A man can dig at the rate of 24 *chi* per day and can make and lay bricks at the rate of 30 per day. If there are 50000 men working together in how many days can they finish the work? What are the sides, the width, and the height of the wall and the depth of the ditch?

Ans. Bottom of the wall:

the exterior side, 10 *li* 10 *bu*;

the interior side, 10 *li*;

the width, 25 *chi*.

Top of the wall:

the exterior side, 10 *li* 6 *bu*;

the interior side, 10 *li* 2 *bu*;

the width, 10 *chi*.

Height of the wall, 36 *chi*;

Depth of the ditch, 25 *chi* $5\frac{483}{2467}$ *cun*;

The work will be finished in $51\frac{473}{1125}$ days.

Process. Let the element *tian* be the width of the wall at the bottom. From the statement we have 412348 for the negative *shi*, $11148\frac{3}{250}$ for the positive *fang*, $14898\frac{1}{250}$ for the positive first *lian*, $131\frac{4}{250}$ for the negative last *lian*,[5] and 1 for the positive *yu*, an expression[6] of the fourth degree whose root is the required width. The other dimensions can be obtained from the addition and subtraction method. The method for finding the depth of the ditch is as follows: Add to three-fourths of the volume four-fifths of the product of the volume of one brick by the number of bricks, for the dividend.[7] Then add to itself one-fourth of the length of the exterior side at the lower base[8] and multiply the sum by 3; add to this product six times the width of the top of the

［4］1块砖的体积为$1\times\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$尺，合用砖体积：$V_2=\frac{1}{8}\times 24000000=3000000$尺^3。（郭）

［5］“步”之下的“六分”、“二分”、“八分”分别即0.6步、0.2步、0.8步。（郭）

［6］开方式的现代形式为：$x^4-131.8x^3+14898.2x^2+11148.6x-412348=0$。（陈）此开方式中的小数陈在新皆表示成分数，今改正。下同，恕不再注。（郭）

［7］实为环池的体积。《九章算术》商功章云：“穿地四为壤五，为坚三。”方城与砖为坚土，环池土为穿土，挖出后为壤土，故筑方城和烧砖所需的土积：$V_3=\frac{4}{3}V+\frac{4}{5}V_2$。（郭）

［8］“身加”，常作“身外加”，是宋元时代人们创造的乘除捷算法之一，乘数是首位为1的2位数时用之，即被乘数本身加上它乘乘数的第2位数，比如，以14乘，便是身外加四。（郭）

［9］记环池的上、下内、外周分别为m_1，M_1，m_2，M_2，取$\sqrt{2}-1.4$，$\pi=3$，则上内周的直径为$\sqrt{2}L_2$，上内周$m_1=3\sqrt{2}L_2=3\times 1.4L_2$。上外周的直径为$\sqrt{2}L_2+2b_1$，上外周$M_1=3(\sqrt{2}L_2+2b_1)=3\times 1.4L_2+6b_1$，池底内周与池底外周相等：$m_2=M_2$，池底外周$M_2=m_1+6b_2=3\sqrt{2}L_2+6b_2$，池底停周$M=\frac{1}{2}(m_2+M_2)$，池上停周$m=\frac{1}{2}(m_1+M_1)$。记环池的深为$h_3$，根据《九章算术》刍童术，其体积为$V_3=\frac{1}{6}[(2m+M)b_1+(2M+m)b_2]h_3$。故以$V_3$为实，以$\frac{1}{6}[(2m+M)b_1+(2M+m)b_2]$为法，便求得环池的深$h_3$。（郭）

［10］记V_1为合用砖数，u，v分别为人日常积和人日烧用砖数，则日数$=\frac{(V+V_2)\times 2}{(u+v)\times 50000}$。（郭）

【今译】

今有要修筑一座方城，其体积为45417600尺。只云：方城下面的外方减10步，其余数开平方，加下广，共得65步。又，上述平方根比上面外

ditch for the exterior circumference of the ditch; add to the interior circumference 6 times the width of the bottom of the ditch for the exterior circumference at the bottom of the ditch. One-half the sum of the interior and exterior circumferences of the bottom of the ditch is the *chi di ting zhou*, and one-half the sum of the interior and exterior circumferences of the top of the ditch is the *chi shang ting zhou*. Multiply the *shang ting zhou* by 2, add to the *di ting zhou*, then multiply by the upper width of the ditch, add this product to the product of twice the *di ting zhou* plus the *shang ting zhou* by the lower width of the ditch, and take one-sixth of the result as divisor. Dividing we have the depth of the ditch. [9] The method of finding the number of days for finishing the work is as follows: Multiply the sum of the volume of the wall and the number of bricks by 2, for the dividend. Multiply the amount of digging done by one man and the number of bricks that one could make by the number of men as the divisor. Dividing we have the number required [10].

【Notes】

[1] Let the square city's interior side at the top be l_1, exterior side at the top L_1, the width at the top a_1, interior side at the bottom l_2, exterior side at the bottom L_2, the width at the bottom a_2, and the height h. The method for the volume of one side is the same as that for *qu chi*. The square city's volume is 4 times the volume of one side. The statement means $V = 4 \times \frac{1}{6}\{[2 \times \frac{1}{2}(l_1 + L_1) + \frac{1}{2}(l_2 + L_2)]a_1 + [2 \times \frac{1}{2}(l_2 + L_2) + \frac{1}{2}(l_1 + L_1)]a_2\}h = 45417600$. (G)

[2] Take *bu* as a unit. That is, $\sqrt{L_2 - 10} + a_2 = 65$, $L_1 - \sqrt{L_2 - 10} = 3546$, $L_1 - l_1 = 4$, $L_1 - l_2 = 6$, $a_2 - a_1 = 3$, $h - a_1 = 5\frac{1}{5}$. (G)

[3] It refers that a *qu chi* in the form of a ring is besides the square city. In fact, its earth is dug for building the city. Its width at the top is $b_1 = 35$, the width at the bottom is $b_2 = 30$. The method for the volume of the ditch is the same as the method for *qu chi* in *The Nine Chapters of Mathematical Procedures*. (G)

方少3546步。上面的内、外方相差4步。上面的外方比下面的内方多6步。上、下广相差3步。上广比高少5步1尺。令紧接方城的四角环绕挖掘一环形圆池，取土筑城以及烧砖包城。环池上广3丈5尺，下广3丈。计料内外城头共用条砖2400万个，每个砖长1尺，阔5寸，厚$2\frac{1}{2}$寸。每人每日应完成的土方积数为24尺，每人每日烧砖及完成包城共30个砖。今派差夫5万人，一齐动工，问：方城的上、下内、外方，上、下广，及高，环池的深，以及完成的日数各为多少？

答：方城的下外方10里10步，内方10里，广2丈5尺；

上外方10里6步，内方10里2步，广1丈；

高3丈6尺；

环池的深2丈5尺$5\frac{483}{2467}$寸；

完成的日数$51\frac{473}{1125}$日。

术：设天元一为下广，以如积方法求其解。得到-412348为常数项，$11148\frac{3}{250}$为一次项系数，$14898\frac{1}{250}$为二次项系数，$-131\frac{4}{250}$为三次项系数，1为最高次项系数，开四次方，得到下广。其他诸事由与下广的关系通过加减法求出。　求环池深之术：列出方城的体积，以4乘之，以3除之，置于上；又以一砖的体积乘合用砖数，以4乘之，以5除之；加于上，作为实。又以方城的下外方身外加四，以3乘之，加6倍的环池上广，为环池外周。又环池内周加6倍的环池下广，为环池的底外周。环池内周与底外周相加，取其$\frac{1}{2}$，为环池的底停周。又，环池的内、外周相加，取其$\frac{1}{2}$，为环池的上停周。上停周的2倍，加底停周，以上广乘之，置于上；又底停周的2倍，加上停周，以下广乘之，加于上，除以6，所得作为法。以法除实，便得环池的深。　求自动工到完工的日数之术：列置方城的体积，加入合用砖数，以2人乘之，作为实。将每人每日应完成的土方积数与每人每日的烧用砖数相加，以总共的差夫数乘之，其得数作为法。实除以法，便得到壕沟深。符合所问。

[4] The volume of one brick is $1 \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ cubic *chi*. The volume of all the bricks: $V_2 = \frac{1}{8} \times 24000000 = 3000000$ cubic *chi*. (G)

[5] 6 *fen*, 2 *fen*, and 8 *fen* following *bu* are 0.6 *bu*, 0.2 *bu* and 0.8 *bu* respectively. (G)

[6] The expression in modern form is the equation:$x^4 - 131.8x^3 + 14898.2x^2 + 11148.6x - 412348 = 0$. (C) The decimal fractions of the equation were all expressed as fractions by Ch'en Tsai Hsin. We correct now and do not annotate in following part. (G)

[7] *Shi* is the ditch's volume. The chapter *shang gong* of *The Nine Chapters of Mathematical Procedures* says:"If *chuan di* is 4, it becomes 5 as *rang tu*, and 3 as *jian tu*." A square city and a brick belong to *chuan tu*. It becomes *rang tu* when it was dug out, Therefore the earth for building the square city and making the bricks is, $V_3 = \frac{4}{3} V + \frac{4}{5} V_2$. (G)

[8] *Shen jia* was ofen written as *shen wai jia*, which was one of the methods for quick calculation of multiplication and division in Song and Yuan dynasties. The method was used when multiplier has two positions and the number of the first position is one. That is, add to multiplicand the number of the second position of the multiplier. For example, when the multiplier is 14, add to multiplicand 4. (G)

[9] Let the ditch's interior circumference at the top be m_1, exterior circumference at the top M_1, interior circumference at the bottom m_2, and exterior circumference at the bottom M_2. Let $\sqrt{2} = 1.4$, $\pi = 3$, then the diameter of the interior circumference at the top is $\sqrt{2} L_2$. The interior circumference at the top $m_1 = 3\sqrt{2} L_2 = 3 \times 1.4L_2$. The diameter of the exterior circumference at the top is $\sqrt{2} L_2 + 2b_1$. The exterior circumference at the top $M_1 = 3(\sqrt{2} L_2 + 2b_1) = 3 \times 1.4L_2 + 6b_1$. The interior circumference at the bottom is equal to the exterior circumference. That is, $m_2 = M_2$. The exterior circumference $M_2 = m_1 + 6b_2 = 3\sqrt{2} L_2 + 6b_2$. *Chi di ting zhou* $M = \frac{1}{2}(m_2 + M_2)$, and *chi shang ting zhou* $m = \frac{1}{2}(m_1 + M_1)$. Let the ditch's depth be h_3. According to the *chu tong* method in *The Nine Chapters of Mathematical Procedures*, $V_3 = \frac{1}{6}[(2m + M) b_1 + (2M + m) b_2] h_3$. Therefore, h_3 can be solved by taking V_3 as *shi*, and $\frac{1}{6}[(2m + M) b_1 + (2M + m) b_2]$ as *fa*. (G)

[10] Let *w* be the number of all the bricks, *u* the amount of digging done by one man and *v* the number of bricks that one could make in one day. Then, the number of the needed days is $\frac{(V + V_2) \times 2}{(u + v) \times 50000}$. (G)

3.

【原文】

今有仰观台一所，计积一万八千五百二十八尺[1]。只云：并上、下袤为实，平方开之，得数，减于上广，不及一丈三尺，却与上、下袤差同，又如高三分之一。上、下广差六尺。[2]欲兴功补为圆台，上下斜长就为圆径。限一日毕役。[3]每人常积二十七尺。问：上、下广、袤及高，大、小四段弧积用徒各几何？

答曰：上广二丈一尺，下广二丈七尺；

上袤二丈八尺，下袤三丈六尺；

高二丈四尺；

二大弧积七千七百二十尺，徒二百八十五人二十七分人之二十五；

二小弧积二千七百二尺，徒一百人二十七分人之二。

术曰：立天元一为台上广，如积求之。得一万八千七百七十四为益实，七百二为益方，三百九十一为从上廉，三十六为益下廉，一为正隅，三乘方开之，[4]得上广。余依加减求之。 求二大弧积术曰：上广减于上弦，余，半之，为上两边各补之广。上袤内加补广，为上两边各补之长。又，下广减于下弦，余，半之，为下两边各补之广。下袤内加补广，为下两边各补之长。倍上长，加下长，以上广乘之，于上。又倍下长，加上长，以下广乘之。加上，以高乘之，如六而一，得二大弧之积。如每人常积除之，得用徒。[5] 求二小弧积及用徒者，如前术入之，即得。合问。

【注释】

[1] 仰观台即《九章算术》商功章的刍童。记仰观台的体积，上、下广、袤，高分别为 V，a_1，a_2，b_1，b_2，h，根据《九章算术》，

$$V=\frac{2}{3}[(2a_1+a_2)b_1+(2a_2+a_1)b_2]h=18528。（郭）$$

3. The volume of an observatory is 18528 *chi*[1]. It is said that the square root of the sum of the lengths of the upper and lower bases is less than the width of the upper base by 13 *chi*, equals the difference between the upper and lower lengths, and is one-third of the length; the difference between the widths of the upper and lower bases is 6 *chi*.[2] It is required to enlarge the observatory into a circular form using the diagonals as the diameters of the bases and to complete the work in one day[3]. The rate of one man's work is 27 *chi*. How many men shall be employed for the work? Find also the widths and lengths of the bases and the heights and volumes of the arcs.

Ans. Upper base:

width, 21 *chi*; length, 28 *chi*.

Lower base:

width, 27 *chi*; length, 36 *chi*.

height, 24 *chi*;

Volume of each of the great arcs:

7720 *chi*;men needed for the work, $285\frac{25}{27}$.

Volume of each of the small arcs:

2702 *chi*; men needed for the work, $100\frac{2}{27}$.

Process. Let the element *tian* be the width of the upper base. From the statement we have 18774 for the negative *shi*, 702 for the negative *fang*, 391 for the positive first *lian*, 36 for the negative last *lian*, and 1 for the positive *yu*, an expression[4] of the fourth degree whose root is the width of the upper base. The other dimensions required can be obtained from the addition and subtraction method. The method for finding the volumes of the two great arcs is as follows:One-half of the difference between the

［2］此即：$a_1-\sqrt{b_1+b_2}=13$，$\sqrt{b_1+b_2}=b_2-b_1$，$\sqrt{b_1+b_2}=\frac{1}{3}h$。$a_2-a_1=6$。（郭）

［3］圆台即《九章算术》之圆亭。其上面的直径即上斜，即上面的对角线，下面的直径即下斜，即下面的对角线。（郭）

［4］开方式的现代形式为：$x^4-36x^3+391x^2-702x-18774=0$。（陈）

［5］上弦即仰观台上面的对角线，亦即以上广为勾，上袤为股的弦，记为 c_1。那么，两边各补之广为 $A_1=\frac{1}{2}(c_1-a_1)$，上两边各补之长为 $B_1=b_1+\frac{1}{2}(c_1-a_1)$。下弦即仰观台下面的对角线，亦即以下广为勾，下袤为股的弦，记为 c_2。那么，两边各补之广为 $A_2=\frac{1}{2}(c_2-a_2)$，上两边各补之长为 $B_2=b_2+\frac{1}{2}(c_2-a_2)$。两大弧之积为 $\frac{1}{6}[(2B_1+B_2)A_1+(2B_2+B_1)A_2]h$。（郭）

【今译】

今有仰观台一所，计其体积为18528尺^3。只云：上、下袤相加，作为实，开平方，其得数比上广少1丈三尺，却与上、下袤之差相等，又等于高的$\frac{1}{3}$。上、下广之差为6尺。现欲兴功，将其补为圆台。仰观台的上、下面的对角线就是圆台的上、下直径。限一日完工。每人一日的工作量是27尺。问：仰观台的上、下广、袤，高，以及大、小四段弧体积、用徒各为多少？

答：仰观台上广2丈1尺，下广2丈7尺；

上袤2丈8尺，下袤3丈6尺；

高2丈4尺；

二大弧体积7720尺，用徒$285\frac{25}{27}$人；

二小弧体积2702尺，用徒$100\frac{2}{27}$人。

术：设天元一为仰观台的上广，以如积方法求其解。得到-18774为常数项，-702为一次项系数，391为二次项系数，-36为三次项系数，

hypotenuse and the width of the upper base is the altitude of the segment of the upper base; and the sum of the altitude of the segment and the length of the upper base is the length of the arc of the upper base. One-half of the difference between the hypotenuse and the width of the lower base is the altitude of the segment of the lower base; and the sum of the altitude of the segment and the length of the lower base is the length of the arc of the lower base. Add to the product of the twice the sum of the arcs of the lower and upper bases by the altitude of the segment of the lower base, the product of twice these same arcs by the altitude of the segment of the upper base; multiply their sum by the height and divide by 6. This gives the volume of the great segment. Divide the volume of the great segment by the rate of work of one person and we have the number of workmen for the segment.[5] The method for finding the small segments is similar to the above.

[Notes]

[1] The observatory is the *chu tong* in the chapter *shang gong* of *The Nine Chapters of Mathematical Procedures*. Let the observatory's volume be V, width of the upper base a_1, width of the lower base a_2, length of the upper base b_1, length of the lower base b_2, and height h. According to *The Nine Chapters of Mathematical Procedures*,

$$V=\frac{2}{3}\left[(2a_1+a_2)b_1+(2a_2+a_1)b_2\right]h=18528. \text{ (G)}$$

[2] That is, $a_1-\sqrt{b_1+b_2}=13$, $\sqrt{b_1+b_2}=b_2-b_1$, $\sqrt{b_1+b_2}=\frac{1}{3}h$. $a_2-a_1=6$. (G)

[3] The enlarged observatory is called *yuan tai*. It is *yuan ting* in *The Nine Chapters of Mathematical Procedures*. The diameter of the upper base is *shang xie*, that is,

1为最高次项系数，开四次方，得到上广。其他诸项由与上广的关系通过加减法求出。 求二大弧体积术：以上广减于上弦，取其余数的$\frac{1}{2}$，为上两边各补之广。上袤加各补之广，为上两边各补之长。又，以下广减于下弦，取其余数的$\frac{1}{2}$，为下两边各补之广。下袤加各补之广，为下两边各补之长。将上两边各补之长的2倍，加下两边各补之长，乘以上两边各补之广，置于上；又将下两边各补之长的2倍，加上两边各补之长，乘以下两边各补之广，其得数加于上，乘以高，除以6，就得到二大弧体积。以每人一日的工作量除二大弧体积，就得到用徒人数。 求二小弧体积及用徒人数，援引上术，即得。符合所问。

4.

【原文】

今有造龙尾堤一所[1]。只云：高多上广二尺，如下广三分之二。高并上广，自乘，不及袤九十六尺。[2]每人日程常积二十九尺。用徒一千八百四十人。限一日役毕。问：堤上、下广及高、袤各几何？

答曰：上广一丈，下广一丈八尺；

高一丈二尺；袤五百八十尺。

术曰：立天元一为堤高，如积求之。得四万二十为益实，二十五为益方，五十二为从上廉，五为益下廉，二为从隅，三乘方开之，[3]得堤高。合问。

the upper diagonal. The diameter of the lower base is *xia xie*, that is, the lower diagonal. (G)

[4] The expression in modern form is the equation: $x^4 - 36x^3 + 391x^2 - 702x - 18774 = 0$. (C)

[5] The upper hypotenuse is the diagonal at the upper base of the observatory. The hypotenuse's *gou* and *gu* are the width and the length at the upper base, respectively. Use c_1 for the hypotenuse. Then, the altitude of the segment of the base is as follows: $A_1 = \frac{1}{2}(c_1 - a_1)$. The length of the hypotenuse of the upper bases: $B_1 = b_1 + \frac{1}{2}(c_1 - a_1)$. The lower hypotenuse is the diagonal at the lower base of the observatory. The hypotenuse's *gou* and *gu* are the width and the length at the lower base, respectively. Use c_2 for the hypotenuse. Then, the altitude of the segment of the base is $A_2 = \frac{1}{2}(c_2 - a_2)$. The length of the hypotenuse of the upper bases: $B_2 = b_2 + \frac{1}{2}(c_2 - a_2)$. The product of the two great arcs is $\frac{1}{6}[(2B_1 + B_2)A_1 + (2B_2 + B_1)A_2]h$. (G)

4. The height of a dike[1] exceeds the width of the upper base by 2 *chi*, and is two-thirds of the width of the lower base; the square of the sum of the height and the width of the upper base is less than the length by 96 *chi*.[2] The construction of the dike is completed by 1840 workmen, and the rate of work of one person is 29 *chi*. And it must be finished in one day. Find the widths of the upper and lower bases, and the height and length of the dike.

Ans. The width of the upper base, 10 *chi*;
the width of the lower base, 18 *chi*;
height, 12 *chi*; length, 580 *chi*.

Process. Let the element *tian* be the height of the dike. From the statement we have 40020 for the negative *shi*, 25 for the negative *fang*, 52 for the

【注释】

［1］龙尾堤即《九章算术》之堤。记其体积、上广、下广、袤、高分别为 V, a_1, a_2, b, h,《九章算术》给出其体积公式：$V=\frac{1}{2}(a_1+a_2)bh$。根据《九章算术》给出的穿地与坚土之率为4 : 3，所需穿地体积为$\frac{2}{3}(a_1+a_2)bh$。（郭）

［2］此即：$h-a_1=2$, $h=\frac{2}{3}a_2$, $b-(a_1+h)^2=96$。（郭）

［3］开方式的现代形式为：$2x^4-5x^3+52x^2-25x-40020=0$。（陈）

【今译】

今要修造一座龙尾堤。只云：其高多于上广2尺，等于下广的$\frac{2}{3}$。高与上广相加，自乘，比袤少96尺。每人每日应完成的标准土方积数为29尺，用徒1840人。限一日工作完毕。问：堤的上、下广，及高、袤各为多少？

答：堤的上广1丈，下广1丈8尺；

高1丈2尺，袤580尺。

术：设天元一为堤高，以如积方法求其解。得到 −40020 为常数项，−25 为一次项系数，52为二次项系数，−5为三次项系数，2为最高次项系数，开四次方，得到堤高。符合所问。

5.

【原文】

今有造仰观台一所。只云：上、下袤差一丈四尺。并上、下广，虚加二，为实，六为从方，一为从隅，平方开之，不及上广八尺。上袤多于上广二分之一。高多下袤七尺。[1] 每日用徒二百二十七人，每人日程常积二十四尺，五日役毕。问：台上、下广、袤及高各几何？

答曰：上袤二丈四尺，下袤三丈八尺；

上广一丈二尺，下广二丈六尺；

positive first *lian*, 5 for the negative last *lian*, and 2 for the positive *yu*, an expression [3] of the fourth degree whose root is the required height.

[Notes]

[1] The dike is *long wei di* in Chinese. It is the dike in *The Nine Chapters of Mathematical Procedures*. Let its volume be V, width of the upper base a_1, width of the lower base a_2, length b, and height h. Its volume in *The Nine Chapters of Mathematical Procedures* is as follows: $V = \frac{1}{2}(a_1 + a_2)bh$. According to *The Nine Chapters of Mathematical Procedures*, the ratio of *chuan di* to *jian tu* is 4 to 3. Therefore, the volume of *chuan di* is $\frac{2}{3}(a_1 + a_2)bh$.(G)

[2] That is, $h - a_1 = 2, h = \frac{2}{3}a_2, b - (a_1 + h)^2 = 96$. (G)

[3] The expression in modern form is the equation: $2x^4 - 5x^3 + 52x^2 - 25x - 40020 = 0$. (C)

5. The difference between the upper and the lower bases of an observatory which is to be constructed is 14 *chi*. If we place the sum of the widths of the upper and the lower bases increased by 2 for the positive *shi*, 6 for the positive *fang*, and 1 for the positive *yu*, we have a quadratic expression whose root is less than the width of the upper base by 8 *chi*; the length of the upper base exceeds the width by one-half of itself; and the height exceeds the length of the lower base by 7 *chi*. [1] The work can be done by 227 men in 5 days,

高四丈五尺。

术曰：立天元一为台之上广，如积求之。得七万七千六百四为益实，一千八百一十三为益方，五百四十六为益上廉，三十一为从下廉，六为从隅，三乘方开之，[2] 即台上广。合问。

【注释】

[1] 记仰观台之上、下广、袤，高分别为 a_1，a_2，b_1，b_2，h，此即：$b_2-b_1=14$，$w^2+6w=(a_1+a_2)+2$，$a_1-w=8$，$b_1-a_1=\frac{1}{2}b_1$，$h-b_2=7$。（郭）

[2] 开方式的现代形式为：$6x^4+31x^3-546x^2-1813x-77604=0$。（陈）

【今译】

今要造仰观台一所。只云：上、下袤相差1丈4尺。上、下广相加，再加2，作为实，6为一次项系数，1为最高次项系数，开平方，其得数比上广少8尺。上袤比上广多上袤的$\frac{1}{2}$。高比下袤多7尺。每日用徒227人，每人一日的标准工作量是24尺，限5日工作完毕。问：仰观台的上、下广、袤及高各为多少？

答：仰观台上袤2丈4尺，下袤3丈8尺；

上广1丈2尺，下广2丈6尺；

高4丈5尺。

术：设天元一为仰观台的上广，以如积方法求其解。得到−77604为常数项，−1813为一次项系数，−546为二次项系数，31为三次项系数，6为最高次项系数，开四次方，即是仰观台的上广。符合所问。

6.

【原文】

今有造圆台一所。只云：并上、下周、高，为实，平方除之，如上周弱半。高与上、下周差同。高多开方数[1] 二分之一。[2] 每日用徒一十九

and the rate of the work of each man is 24 *chi*. Find the lengths and the widths of the upper and the lower bases and the height of the observatory.

Ans. Upper base:

length, 24 *chi*; width, 12 *chi*.

Lower base:

length, 38 *chi*; width, 26 *chi*;.

height, 45 *chi*.

Process. Let the element *tian* be the width of the upper base. From the statement we have 77604 for the negative *shi*, 1813 for the negative *fang*, 546 for the negative first *lian*, 31 for the positive last *lian*, and 6 for the positive *yu*, an expression [2] of the fourth degree whose root is the required width.

[Notes]

[1] Let the observatory's width of the upper base be a_1, width of the lower base a_2, length of the upper base b_1, length of the lower base b_2, and the height h. That is, $b_2 - b_1 = 14$, $w^2 + 6w = (a_1 + a_2) + 2$, $a_1 - w = 8$, $b_1 - a_1 = \frac{1}{2} b_1$, $h - b_2 = 7$. (G)

[2] The expression in modern form is the equation: $6x^4 + 31x^3 - 546x^2 - 1813x - 77604 = 0$. (C)

6. A round terrace is to be constructed, such that the square root of the sum of the circumferences of the upper and the lower bases and the height equals the weak-half of the circumference of the upper base; the height of the terrace equals the difference between the circumferences of the upper and the lower

人，限一十二日役毕。每人日程常积三十二尺。问：台高及上、下周各得几何？

答曰：上周四丈八尺，下周七丈二尺，高二丈四尺。

术曰：立大元一为开方数，如积求之。得一十三万一千三百二十八为益实，二十八为从二廉，八为益三廉，一为正隅，四乘方开之，[3] 得十二为开方数。倍之，即高。余依加减求之。合问。

【注释】

[1] 开方数即上、下周，高的平方根。(陈)

[2] 此圆台即《九章算术》的圆亭，记其上、下周，高分别为 l_1，l_2，h。《九章算术》给出其体积公式为 $V=\frac{1}{36}(l_1l_2+l_1^2+l_2^2)h$。《四元玉鉴》使用《九章算术》的公式。此条题设即：$\sqrt{l_1+l_2+h}=\frac{1}{4}l_1$，$h=l_2-l_1$，$h-\sqrt{l_1+l_2+h}=\frac{1}{2}h$。(郭)

[3] 开方式的现代形式为：$x^5-8x^4+28x^3-131328=0$。(陈)

【今译】

今要造圆台一所。只云：上、下周与高相加，作为实，开平方，其得数是上周的 $\frac{1}{4}$。高等于上、下周之差。高比开方得数多其 $\frac{1}{2}$。每日用徒19人，限12日完工，每人一日的标准工作量是32尺。问：圆台的高及上、下周各为多少？

答：上周4丈8尺，下周7丈2尺，高2丈4尺。

术：设天元一为开方得数，以如积方法求其解。得到 −131328 为常数项，28为三次项系数，−8为四次项系数，1为最高次项系数，开五次方，得12为开方得数。加倍，就是高。其他诸项由与台高的关系通过加减法求出。符合所问。

bases and exceeds the square root [1] by one-half of itself.[2] The work must be finished by 19 men in 12 days; the rate of work of one person is 32 *chi*. Find the height and the circumferences of the upper and lower bases.

Ans. The circumference of the upper base, 48 *chi*;

the circumference of the lower base, 72 *chi*;

height, 24 *chi*.

Process. Let the element *tian* be the square root of the sum of the circumferences of the upper and lower bases and the height of the terrace. From the statement we have 131328 for the negative *shi*, 28 for the positive second *lian*, 8 for the negative third *lian*, and 1 for the positive *yu*, an expression [3] of the fifth degree whose root is 12. The twice of the root is the height. The other dimensions can be found from the addition and subtraction method.

[Notes]

[1] The square root of the sum of the circumferences of the upper and lower bases and the height. (C)

[2] The round terrace is *yuan ting* in *The Nine Chapters of Mathematical Procedures*. Let its circumference of the upper base be l_1, circumference of the lower base l_2, and height h. Its volume formula in *The Nine Chapters of Mathematical Procedures* is as follows: $V = \frac{1}{36}(l_1 l_2 + l_1^2 + l_2^2)h$. The *Jade Mirror of the Four Unknowns* uses the formula. The statement is $\sqrt{l_1 + l_2 + h} = \frac{1}{4}l_1, h = l_2 - l_1, h - \sqrt{l_1 + l_2 + h} = \frac{1}{2}h$. (G)

[3] The expression in modern form is the equation: $x^5 - 8x^4 + 28x^3 - 131328 = 0$. (C)

7.

【原文】

今有造方台一所[1]，共支功、食钱二百五十七贯六百二十二文七分文之六。只云：以台高为正实，十为益方，一为正隅，平方开之。所得再为实，开平方除之，少如先开方数中半。上方多如先开方数强半。上、下方和得三十八尺。[2]每人日程常积二十八尺。每三人支钱二贯四百七十七文七分文之一。用徒日自倍，令四日役毕。问：台上、下方，高，及逐日用徒、支钱各几何？

答曰：上方一丈六尺，下方二丈二尺，高二丈四尺；

初日二十人五分人之四，钱一十七贯一百七十四文七分文之六；

次日四十一人五分人之三，钱三十四贯三百四十九文七分文之五；

三日八十三人五分人之一，钱六十八贯六百九十九文七分文之三；

末日一百六十六人五分人之二，钱一百三十七贯三百九十八文七分文之六。

术曰：立天元一为后开方数，如积求之。得六千五百五十二为益实，三千六百一十为从上廉，七百四十一为益三廉，七十八为从五廉，四为益隅，七乘方开之，[3]得二尺为后开方数。自之，即先开方数。四之，为台上方。余依加减求之。每日用徒及钱者，如法求之。合问。

【注释】

[1] 此方台即《九章算术》的方亭，记其上、下方，高分别为 a、b、h，《九章算术》给出其体积公式为：$V=\frac{1}{3}(ab+a^2+b^2)h$。《四元玉鉴》使用《九章算术》的公式。（郭）

7. The construction of a square terrace will cost $257622\frac{6}{7}$ cash[1]. If we place the height of the terrace for the positive *shi*, 10 for the negative *fang*, and 1 for the positive *yu*, the square root of the root of this quadratic expression is less than the latter by one-half of itself; a side of the upper base exceeds the root of the expression by its strong-half; the sum of a side of the upper base and a side of the lower base is 38 *chi*[2]. The rate of work of one person is 28 *chi*, and the rate of pay for three persons is $2477\frac{1}{7}$ cash. The number of workmen is increased each day in a geometrical progression whose ratio is 2. The work must be completed in four days. Find the height and the sides of the bases, the number of the workmen employed each day, and the rate of pay per day.

Ans. A side of the upper base, 16 *chi*; a side of the lower base, 22 chi; height, 24 *chi*.

First day, workman employed, $20\frac{4}{5}$.

payment, $17174\frac{6}{7}$ cash;

Second day, workman employed, $41\frac{3}{5}$.

payment, $34349\frac{5}{7}$ cash;

Third day, workman employed, $83\frac{1}{5}$.

payment, $68699\frac{3}{7}$ cash;

Last day, workman employed, $166\frac{2}{5}$.

payment, $137398\frac{6}{7}$ cash.

Process. Let the element *tian* be the square root of the root of the given expression. From the statement we have 6552 for the negative *shi*, 3610 for the positive first *lian*, 741 for the negative third *lian*, 78 for the positive

[2] 此条题设即：记先开方数为 w，它由开方式 $w^2-10w+h=0$ 求出。$w-\sqrt{w}=\frac{1}{2}w$。$a-w=\frac{3}{4}a$，$a+b=38$。$\sqrt{w}$ 为后开方数。（郭）

[3] 开方式的现代形式为：$-4x^8+78x^6-741x^4+3610x^2-6552=0$。（陈）

【今译】

今要造方台一所，共支付工钱及饭钱257贯$622\frac{6}{7}$文。只云：以台高作为实，−10为一次项系数，1为最高次项系数，开平方。再以其得数作为实，开平方，其得数比先开方得数少先开方得数的$\frac{1}{2}$。上方比先开方得数多上方的$\frac{3}{4}$。上、下方之和为38。每人一日的标准工作量是28尺。每3人支钱2贯$477\frac{1}{7}$文。用徒逐日加倍，限4日完工，问：方台的上、下方，高，及逐日用徒、支钱各为多少？

答：上方1丈6尺，下方2丈2尺，高2丈4尺；

第1日$20\frac{4}{5}$人，17贯$174\frac{6}{7}$文钱；

第2日$41\frac{3}{5}$人，34贯$349\frac{5}{7}$文钱；

第3日$83\frac{1}{5}$人，68贯$699\frac{3}{7}$文钱；

末日$166\frac{2}{5}$人，137贯$398\frac{6}{7}$文钱。

术：设天元一为后开方得数，以如积方法求其解。得到−6552为常数项，3610为二次项系数，−741为四次项系数，78为六次项系数，−4为最高次项系数，开八次方，得到2尺，为后开方得数。后开方得数自乘，就是先开方得数。以4乘之，就是台的上方。下方及高由与台的上方的关系通过加减法求出。每日用徒及支钱者，按照已有的方法求之，便符合所问。

fifth *lian*, and 4 for the negative *yu*, an expression [3] of the eighth degree whose root is 2. The square of this root is the root of the given expression, and 4 times the root of the given expression is a side of the upper base of the terrace. The number of workmen employed and the rate of pay per day can be found according to the rule.

【Notes】

[1] The square terrace is *fang ting* in *The Nine Chapters of Mathematical Procedures*. Let its side of the upper base be a, side of the lower base b, and height h. Its volume formula given by *The Nine Chapters of Mathematical Procedures* is as follows: $V = \frac{1}{3}(ab + a^2 + b^2)h$. The *Jade Mirror of the Four Unknowns* uses the formula. (G)

[2] What the statement means is as follows: Let the the number of former square root be w which can be extracted by the equation $w^2 - 10w + h = 0$. $w - \sqrt{w} = \frac{1}{2}w$, $a - w = \frac{3}{4}a$. $a + b = 38$. $\sqrt{w}$ is the number of latter square root. (G)

[3] The expression in modern form is the equation: $-4x^8 + 78x^6 - 741x^4 + 3610x^2 - 6552 = 0$. (C)

和分索隐　一十三问

1.

【原文】

今有勾三步十分步之九，股五步五分步之一。[1] 问：弦几何？

答曰：六步二分步之一。

术曰：立天元一为弦，如积求之。得一十万五千六百二十五为益实，二千五百为从隅，平方开之，[2] 得弦。不尽，按连枝同体术求之。合问。

【注释】

[1] 记勾股形的勾、股、弦分别为 a，b，c，此即：$a = 3\frac{9}{10}$，$b = 5\frac{1}{5}$。（郭）

[2] 开方式的现代形式为：$2500x^2 - 105625 = 0$。（陈）

【今译】

今有勾 $3\frac{9}{10}$ 步，股 $5\frac{1}{5}$ 步。问：弦为多少？

答：$6\frac{1}{2}$ 步。

术：设天元一为弦，以如积方法求其解。得到 -105625 为常数项，2500 为最高次项系数，开平方，得到弦。开方不尽，按照连枝同体术求其分数部分。符合所问。

2.

【原文】

今有股五步五分步之一，弦六步二分步之一。[1] 问：勾几何？

答曰：三步十分步之九。

术曰：立天元一为勾，如积求之。得一千五百二十一为益实，一百为从隅，平方开之，得勾。不尽，按之分法求之。合问。[2]

【注释】

[1] 此即：$b = 5\frac{1}{5}$，$c = 6\frac{1}{2}$。（郭）

He Fen Suo Yin (Problems on Equations with Fractional Roots)

13 Problems

1. The *gou* is $3\frac{9}{10}$ *bu* and the *gu* $5\frac{1}{5}$ *bu*.[1] Find the *xian*.

Ans. $6\frac{1}{2}$ *bu*.

Process. Let the element *tian* be *xian*. From the statement we have 105625 for the negative *shi* and 2500 for the positive *yu*, a quadratic expression[2] which solved by the *lian zhi tong ti* method gives the required *xian*.

[Notes]

[1] Let the right triangle's *gou* be a, *gu* b, *xian* c. That is, $a = 3\frac{9}{10}$, $b = 5\frac{1}{5}$. (G)

[2] The expression in modern form is the equation: $2500x^2 - 105625 = 0$. (C)

2. The *gu* is $5\frac{1}{5}$ *bu*, and the *xian* is $6\frac{1}{2}$ *bu*.[1] Find the *gou*.

Ans. $3\frac{9}{10}$ *bu*.

Process. Let the element *tian* be *gou*. From the statement we have 1521 for the negative *shi* and 100 for the positive *yu*, a quadratic expression. Solving by the *zhi fen* method we have the required *gou*.[2]

[Notes]

[1] That is, $b = 5\frac{1}{5}$, $c = 6\frac{1}{2}$. (G)

[2]开方式的现代形式为：$100x^2 - 1521 = 0$。(陈)之分法即连枝同体术。(郭)

【今译】

今有股$5\frac{1}{5}$步，弦$6\frac{1}{2}$步。问：勾为多少？

答：$3\frac{9}{10}$步。

术：设天元一为勾，以如积方法求其解。得到-1521为常数项，100为最高次项系数，开平方，得到勾。开方不尽，按照之分法求其分数部分。符合所问。

3.

【原文】

今有弦六步二分步之一，勾三步十分步之九。[1] 问：股几何？

答曰：五步五分步之一。

术曰：立天元一为股，如积求之。得二千七百四为益实，一百为从隅，平方开之，[2] 得股。不尽，按之分法入之。合问。

【注释】

[1] 此即：$c = 6\frac{1}{2}$，$a = 3\frac{9}{10}$。(郭)

[2] 开方式的现代形式为：$100x^2 - 2704 = 0$。(陈)

【今译】

今有弦$6\frac{1}{2}$步，勾$3\frac{9}{10}$步。问：股为多少？

答：$5\frac{1}{5}$步。

术：设天元一为股，以如积方法求其解。得到-2704为常数项，100为最高次项系数，开平方，得到股。开方不尽，按照之分法求其分数部分。符合所问。

4.

【原文】

今有直积一十八步一十二分步之五。[1] 只云：长取四分之三，阔取三分之一，为共，如长一十七分之一十六。[2] 问：长、平各几何？

[2] The expression in modern form is the equation: $100x^2 - 1521 = 0$. (C) The *zhi fen* method is also called *lian zhi tong ti* method. (G)

3. Given the *xian* $6\frac{1}{2}$ *bu* and the *gou* $3\frac{9}{10}$ *bu*.[1] Find the *gu*.

Ans. $5\frac{1}{5}$ *bu*.

Process. Let the element *tian* be *gu*. From the statement we have 2704 for the negative *shi* and 100 for the positive *yu*, a quadratic expression[2]. Solving by the *zhi fen* method we have the required *gu*.

[Notes]

[1] That is, $c = 6\frac{1}{2}$, $a = 3\frac{9}{10}$. (G)

[2] The expression in modern form is the equation: $100x^2 - 2704 = 0$. (C)

4. The *zhi ji* is $18\frac{5}{12}$ *bu*;[1] and the sum of three-fourths of the length and one-third of the width is $\frac{16}{17}$ of the length.[2] Find the length and the width.

Ans. Length, $5\frac{2}{3}$ *bu*; width, $3\frac{1}{4}$ *bu*.

答曰：长五步三分步之二，阔三步四分步之一。

术曰：立天元一为长，如积求之。得三千七百五十七为益实，一百一十七为从隅，平方开之，[3] 得长。不尽，按之分法求之。合问。

【注释】

[1] 记直积的面积、长、阔分别为 S, b, a，此即：$S = ab = 18\frac{5}{12}$。（郭）

[2] "一十七分之"原文讹作"一十七分步之"，今以意校正。此即：$\frac{3}{4}b + \frac{1}{3}a = \frac{16}{17}b$。（郭）

[3] 开方式的现代形式为：$117x^2 - 3757 = 0$。（陈）

【今译】

今有直积$18\frac{5}{12}$步。只云：取长的$\frac{3}{4}$，阔的$\frac{1}{3}$，二者之和为长的$\frac{16}{17}$。问：长、平各为多少？

答：长$5\frac{2}{3}$步，阔$3\frac{1}{4}$步。

术：设天元一为长，以如积方法求其解。得到 -3757 为常数项，117 为最高次项系数，开平方，得到长。开方不尽，按照之分法求其分数部分。符合所问。

5.

【原文】

今有直积一百一十步二分步之一[1]。只云：长、平和二十一步一十二分步之五[2]。问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为平，如积求之。得一千三百二十六为益实，二百五十七为从方，一十二为益隅，平方开之，[3] 得平。不尽，按之分法求之。合问。

【注释】

[1] 此即：$S = ab = 110\frac{1}{2}$。（郭）

[2] 此即：$a + b = 21\frac{5}{12}$。（郭）

[3] 开方式的现代形式为：$-12x^2 + 257x - 1326 = 0$。（陈）

Process. Let the element *tian* be the length. From the statement we have 3757 for the negative *shi* and 117 for the positive *yu*, a quadratic expression [3]. Solving by the *zhi fen* method we have the required length.

[Notes]

[1] Let *zhi ji*'s area be S, length b, width a. That is, $S = ab = 18\frac{5}{12}$. (G)

[2] *Shi qi fen zhi* was mistaken for *shi qi fen bu zhi* in the original text. According to the statement's meaning, I revise it. That is, $\frac{3}{4}b + \frac{1}{3}a = \frac{16}{17}b$. (G)

[3] The expression in modern form is the equation: $117x^2 - 3757 = 0$. (C)

5. The *zhi ji* is $110\frac{1}{2}$ *bu*;[1] the sum of the length and the width is $21\frac{5}{12}$ *bu*[2]. Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the width. From the statement we have 1326 for the negative *shi*, 257 for the positive *fang*, and 12 for the negative *yu*, a quadratic expression [3]. Solving by the *zhi fen* method we have the required width.

[Notes]

[1] That is, $S = ab = 110\frac{1}{2}$. (G)

【今译】

今有直积 $110\frac{1}{2}$ 步。只云：长、平之和为 $21\frac{5}{12}$ 步。问：长、平各为多少？

答：平 $8\frac{2}{3}$ 步，长 $12\frac{3}{4}$ 步。

术：设天元一为平，以如积方法求其解。得到 -1326 为常数项，257 为一次项系数，-12 为最高次项系数，开平方，得到平。开方不尽，按照之分法求其分数部分。符合所问。

6.

【原文】

今有直积一百一十步二分步之一。只云：长、平差四步一十二分步之一。[1] 问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为平，如积求之。得一千三百二十六为益实，四十九为从方，一十二为从隅，平方开之，[2] 得平。不尽，按之分法求之。合问。

【注释】

[1] 此即：$S=ab=110\frac{1}{2}$，$b-a=4\frac{1}{12}$。（郭）

[2] 开方式的现代形式为：$12x^2+49x-1326=0$。（陈）

【今译】

今有直积 $110\frac{1}{2}$ 步。只云：长、平之差为 $4\frac{1}{12}$ 步。问：长、平各为多少？

答：平 $8\frac{2}{3}$ 步，长 $12\frac{3}{4}$ 步。

术：设天元一为平，以如积方法求其解。得到 -1326 为常数项，49 为一次项系数，12 为最高次项系数，开平方，得到平。开方不尽，按照之分法求其分数部分。符合所问。

[2] That is, $a + b = 21\frac{5}{12}$. (G)

[3] The expression in modern form is the equation: $-12x^2 + 257x - 1326 = 0$. (C)

6. The *zhi ji* is $110\frac{1}{2}$ *bu*, and the difference between the length and the width is $4\frac{1}{12}$ *bu*.[1] Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the width. From the statement we have 1326 for the negative *shi*, 49 for the positive *fang*, and 12 for the positive *yu*, a quadratic expression [2]. Solving by the *zhi fen* method we have the required width.

[Notes]

[1] That is, $S = ab = 110\frac{1}{2}$, $b - a = 4\frac{1}{12}$. (G)

[2] The expression in modern form is the equation: $12x^2 + 49x - 1326 = 0$. (C)

7.

【原文】

今有直积一百一十步二分步之一。只云：三平内减二长，余有六分步之三。[1] 问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为长，如积求之。得一千九百八十九为益实，三为从方，一十二为从隅，平方开之，[2] 得长。不尽，按之分法入之。合问。

【注释】

[1] 此即：$S=ab=110\frac{1}{2}$，$3a-2b=\frac{3}{6}$。（郭）

[2] 开方式的现代形式为：$12x^2+3x-1989=0$。（陈）

【今译】

今有直积 $110\frac{1}{2}$ 步。只云：平的 3 倍，减去长的 2 倍，余 $\frac{3}{6}$ 步。问：长、平各为多少？

答：平 $8\frac{2}{3}$ 步，长 $12\frac{3}{4}$ 步。

术：设天元一为长，以如积方法求其解。得到 -1989 为常数项，3 为一次项系数，12 为最高次项系数，开平方，得到长。开方不尽，按照之分法求其分数部分。符合所问。

8.

【原文】

今有直积一百一十步二分步之一。只云：平取八分之三，长取九分之四，共得八步一十二分步之一十一。[1] 问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为长，如积求之。得三万五千八百二为益实，七千七百四为从方，三百八十四为益隅，平方开之，[2] 得长。不尽，按之分法求之。合问。

【注释】

[1] 此即：$S=ab=110\frac{1}{2}$，$\frac{3}{8}a+\frac{4}{9}b=8\frac{11}{12}$。（郭）

7. The *zhi ji* is $110\frac{1}{2}$ *bu*. If from 3 times the width we subtract twice the length the remainder is $\frac{3}{6}$ *bu*.[1] Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the length. From the statement we have 1989 for the negative *shi*, 3 for the positive *fang*, and 12 for the positive *yu*, a quadratic expression[2]. Solving by the *zhi fen* method we have the required length.

【 Notes 】

[1] That is, $S = ab = 110\frac{1}{2}$, $3a - 2b = \frac{3}{6}$. (G)

[2] The expression in modern form is the equation: $12x^2 + 3x - 1989 = 0$. (C)

8. The *zhi ji* is $110\frac{1}{2}$ *bu*; the sum of $\frac{3}{8}$ of the width and $\frac{4}{9}$ of the length is $8\frac{11}{12}$ *bu*.[1] Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the length. From the statement we have 35802 for the negative *shi*, 7704 for the positive *fang*, and 384 for the negative *yu*, a quadratic expression[2]. Solving by the *zhi fen* method we have the required length.

［2］开方式的现代形式为：$-384x^2+7704x-35802=0$。（陈）

【今译】

今有直积$110\frac{1}{2}$步。只云：取平的$\frac{3}{8}$，长的$\frac{4}{9}$，二者之和为$8\frac{11}{12}$步。问：长、平各为多少?

答：平$8\frac{2}{3}$步，长$12\frac{3}{4}$步。

术：设天元一为长，以如积方法求其解。得到-35802为常数项，7704为一次项系数，-384为最高次项系数，开平方，得到长。开方不尽，按照之分法求其分数部分。符合所问。

9.

【原文】

今有直积一百一十步二分步之一。只云：并一长、二平、三和、四较，共得一百一十步一十二分步之八。[1] 问：长、平各几何?

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为平，如积求之。得一万六百八为益实，一千三百二十八为从方，一十二为益隅，平方开之，[2] 得平。不尽，按之分术求之。合问。

【注释】

［1］此即：$S=ab=110\frac{1}{2}$，$b+2a+3(a+b)+4(b-a)=110\frac{8}{12}$。（郭）

［2］开方式的现代形式为：$-12x^2+1328x-10608=0$。（陈）

【今译】

今有直积$110\frac{1}{2}$步。只云：长的1倍，平的2倍，长、平和的3倍，长、平差的4倍，共得$110\frac{8}{12}$步。问：长、平各为多少?

答：平$8\frac{2}{3}$步，长$12\frac{3}{4}$步。

术：设天元一为平，以如积方法求其解。得到-10608为常数项，1328为一次项系数，-12为最高次项系数，开平方，得到平。开方不尽，按照之分术其求分数部分。符合所问。

【 Notes 】

[1] That is, $S = ab = 110\frac{1}{2}$, $\frac{3}{8}a + \frac{4}{9}b = 8\frac{11}{12}$. (G)

[2] The expression in modern form is the equation: $-384x^2 + 7704x - 35802 = 0$. (C)

9. The *zhi ji* is $110\frac{1}{2}$ *bu*; the sum of the length, two times the width, three times the *he*, and four times the *jiao*, is $110\frac{8}{12}$ *bu*.[1] Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the width. From the statement we have 10608 for the negative *shi*, 1328 for the positive *fang*, and 12 for the negative *yu*, a quadratic expression[2]. Solving by the *zhi fen* method we have the required width.

【 Notes 】

[1] That is, $S = ab = 110\frac{1}{2}$, $b + 2a + 3(a + b) + 4(b - a) = 110\frac{8}{12}$. (G)

[2] The expression in modern form is the equation: $-12x^2 + 1328x - 10608 = 0$. (C)

10.

【原文】

今有直积，加平四分之一，共得一百一十二步三分步之二。[1] 只云：一和、五平内减四长、三较，余一步一十二分步之六。[2] 问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为平，如积求之。得二千二十八为益实，二十七为从隅，平方开之，[3] 得平。不尽，按之分法求之。合问。

【注释】

[1] 此即：$ab+\frac{1}{4}a=112\frac{2}{3}$。（郭）

[2] 此即：$[(a+b)+5a]-[4b+3(b-a)]=1\frac{6}{12}$。（郭）

[3] 开方式的现代形式为：$27x^2-2028=0$。（陈）

【今译】

今有直积，加平的$\frac{1}{4}$，共得$112\frac{2}{3}$步。只云：长、平和的1倍与平的5倍，减去长的4倍与长平差的3倍，余为$1\frac{6}{12}$步。问：长、平各为多少？

答：平$8\frac{2}{3}$步，长$12\frac{3}{4}$步。

术：设天元一为平，以如积方法求其解。得到-2028为常数项，27为最高次项系数，开平方，得到平。开方不尽，按照之分法求其分数部分。符合所问。

11.

【原文】

今有直积，加平，减较，余一百一十五步一十二分步之一。[1] 只云：三长、二平多于二和、三较一十二分步之六。[2] 问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为长，如积求之。得四千一百三十一为益实，一十八为从方，二十四为从隅，平方开之，[3] 得长。不尽，按之分法求之。合问。

10. The sum of the *zhi ji* and one-fourth of the width is $112\frac{2}{3}$ *bu*;[1] the sum of the *he* and 5 times the width, exceeds the sum of 4 times the length and 3 times the *jiao* by $1\frac{6}{12}$ *bu*.[2] Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the width. From the statement we have 2028 for the negative *shi*, and 27 for the positive *yu*, a quadratic expression[3]. Solving by the *zhi fen* method we have the required width.

[Notes]

[1] That is, $ab+\frac{1}{4}a=112\frac{2}{3}$. (G)

[2] That is, $[(a+b)+5a]-[4b+3(b-a)]=1\frac{6}{12}$. (G)

[3] The expression in modern form is the equation: $27x^2-2028=0$. (C)

11. The *zhi ji* plus the width minus the *jiao* is equal to $115\frac{1}{12}$ *bu*;[1] the sum of 3 times the length and 2 times the width exceeds the sum of 2 times the *he* and 3 times the *jiao* $\frac{6}{12}$ *bu*.[2] Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the length. From the statement we have 4131 for the negative *shi*, 18 for the positive *fang*, and 24 for the positive

【注释】

[1] 此即：$ab + a - (b - a) = 115\frac{1}{12}$。（郭）

[2] 此即：$(2a + 3b) - [2(a + b) + 3(b - a)] = \frac{6}{12}$。（郭）

[3] 开方式的现代形式为：$24x^2 + 18x - 4131 = 0$。（陈）

【今译】

今有直积，加平，减去长、平差，余 $115\frac{1}{12}$ 步。只云：长的3倍与平的2倍比长平和的2倍与长平差的3倍多 $\frac{6}{12}$ 步。问：长、平各为多少？

答：平$8\frac{2}{3}$步，长$12\frac{3}{4}$步。

术：设天元一为长，以如积方法求其解。得到 -4131 为常数项，18为一次项系数，24为最高次项系数，开平方，得到长。开方不尽，按照之分法求其分数部分。符合所问。

12.

【原文】

今有直积，加长，以平乘之，得一千六十八步六分步之一。[1] 只云：二和、一长内减三平、五较，余九步六分步之一。[2] 问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为平，如积求之。得一万二千八百一十八为益实，五十五为益方，三十一为益廉，二十四为从隅，立方开之，[3] 得平。不尽，按之分法求之。合问。

【注释】

[1] 此即：$(ab + b)a = 1068\frac{1}{6}$。（郭）

[2] 此即：$[2(a + b) + b] - [3a + 5(b - a)] = 9\frac{1}{6}$。（郭）

[3] 开方式的现代形式为：$24x^3 - 31x^2 - 55x - 12818 = 0$。（陈）

yu, a quadratic expression [3]. Solving by the *zhi fen* method we have the required length.

【 Notes 】

[1] That is, $ab + a - (b - a) = 115\frac{1}{12}$. (G)

[2] That is, $(2a + 3b) - [2(a + b) + 3(b - a)] = \frac{6}{12}$. (G)

[3] The expression in modern form is the equation: $24x^2 + 18x - 4131 = 0$. (C)

12. The sum of the *zhi ji* and the length multiplied by the width is equal to $1068\frac{1}{6}$ *bu*;[1] the sum of 2 times the *he* and the length less the sum of 3 times the width and 5 times the *jiao* is $9\frac{1}{6}$ *bu*.[2] Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the width. From the statement we have 12818 for the negative *shi*, 55 for the negative *fang*, 31 for the negative *lian*, and 24 for the positive *yu*, a cubic expression [3]. Solving by the *zhi fen* method we have the required width.

【 Notes 】

[1] That is, $(ab + b)a = 1068\frac{1}{6}$. (G)

[2] That is, $[2(a + b) + b] - [3a + 5(b - a)] = 9\frac{1}{6}$. (G)

【今译】

今有直积，加长，以平乘之，得$1068\frac{1}{6}$步。只云：长、平和的2倍与长的1倍之和，减去平的3倍与长平差的5倍之和，余为$9\frac{1}{6}$步。问：长、平各为多少？

答：平$8\frac{2}{3}$步，长$12\frac{3}{4}$步。

术：设天元一为平，以如积方法求其解。得到-12818为常数项，-55为一次项系数，-31为二次项系数，24为最高次项系数，开立方，得到平。开方不尽，按照之分法求其分数部分。符合所问。

13.

【原文】

今有直积，自乘，减和幂，余一万一千七百五十一步一百四十四分步之八十三。[1]只云：较不及平四步一十二分步之七[2]。问：长、平各几何？

答曰：平八步三分步之二，长一十二步四分步之三。

术曰：立天元一为平，如积求之。得一百六十九万五千二百五十二为益实，三千九百六十为从方，一千七百二十九为从上廉，二千六百四十为益下廉，五百七十六为从隅，三乘方开之，[3]得平。不尽，按之分法求之。再得一百四万二千八十四亿五千二百八十一万二千八百为益实，二千三百三十七亿三十六万一百九十二为从方，九千一百九十万二千五百二十八为从上廉，一万五千七百九十二为从下廉，一为正隅，三乘方开之，[4]得三百八十四，与分母约之。合问[5]。

【注释】

[1] 此即：$(ab)^2-(a+b)^2=11751\frac{83}{144}$。（郭）

[2] 此即：$a-(b-a)=4\frac{7}{12}$。（郭）

[3] 开方式的现代形式为：$576x^4-2640x^3+1729x^2+3960x-1695252=0$。（陈）

[4] 开方式的现代形式为：$y^4+15792y^3+91902528y^2+233700360192y-104208452812800=0$。（陈）

[3] The expression in modern form is the equation: $24x^3 - 31x^2 - 55x - 12818 = 0$. (C)

13. The difference between the squares of the *zhi ji* and the *he* is $11751\frac{83}{144}$ *bu*;[1] the *jiao* is less than the width by $4\frac{7}{12}$ *bu*[2]. Find the length and the width.

Ans. Width, $8\frac{2}{3}$ *bu*; length, $12\frac{3}{4}$ *bu*.

Process. Let the element *tian* be the width. From the statement we have 1695252 for the negative *shi*, 3960 for the positive *fang*, 1729 for the positive first *lian*, 2640 for the negative last *lian*, and 576 for the positive *yu*, an expression[3] of the fourth degree. Solving we have the integral part of the root. The fractional part can be obtained by applying the *zhi fen* method by which we have the following expression: 104208452812800 for the negative *shi*, 233700360192 for the positive *fang*, 91902528 for the positive first *lian*, 15792 for the positive last *lian*, and 1 for the positive *yu*, an expression[4] of the fourth degree whose root is 384. After reducing the fraction to its lowest terms we have the whole root required[5].

【Notes】

[1] That is, $(ab)^2 - (a+b)^2 = 11751\frac{83}{144}$. (G)

[2] That is, $a - (b-a) = 4\frac{7}{12}$. (G)

[5] 过程如下：

576	−2640	+1729	+3960	−1695252 (8
	4608	+15744	+139784	+1149952
	1968	17473	143744	545300
	4608	52608	560648	
	6576	70081	704392	
	4608	89472		
	11184	159553		
	4608			
	15792			

余式可变形为 $576x^4 + 15792x^3 + 159553x^2 + 704392x - 545300 = 0$, (1)

之分法与现代方法中将一个方程变形于另外一个方程，且新方程的根为原方程根的 m 倍的方法相同。因此，在解此题时，以576乘表达式(1)的第二项，以576^2乘第三项，以576^3乘第四项，等等。为了化简问题，以576除此表达式，那么我们得到：

$y^4 + 15792y^3 + 91902528y^2 + 233700360192y - 104208452812800 = 0$

解这个表达式，我们得到384，其为所求根的576倍。故，$\frac{384}{576}$ 或者 $\frac{2}{3}$ 是原方程根的分数部分。(陈)

【今译】

今有直积，自乘，减长、平和之幂，余 $11751\frac{83}{144}$ 步。只云：长、平差比平少 $4\frac{7}{12}$ 步。问：长、平各为多少？

答：平 $8\frac{2}{3}$ 步，长 $12\frac{3}{4}$ 步。

术：设天元一为平，以如积方法求其解。得到−1695252为常数项，3960为一次项系数，1729为二次项系数，−2640为三次项系数，576为最高次项系数，开四次方，得到平。开方不尽，按照之分法求分数部分：再得到−104208452812800为常数项，233700360192为一次项系数，91902528为二次项系数，15792为三次项系数，1为最高次项系数，开四次方，得到384，作为分子，与分母576相约减，便得到分数部分。符合所问。

[3] The expression in modern form is the equation: $576x^4 - 2640x^3 + 1729x^2 + 3960x - 1695252 = 0$. (C)

[4] The expression in modern form is the equation: $y^4 + 15792y^3 + 91902528y^2 + 233700360192y - 104208452812800 = 0$. (C)

[5] The process is as follows:

576	−2640	+1729	+3960	−1695252 (8
	4608	+15744	+139784	+1149952
	1968	17473	143744	545300
	4608	52608	560648	
	6576	70081	704392	
	4608	89472		
	11184	159553		
	4608			
	15792			

Forming the remainders into a new expression we have

$576x^4 + 15792x^3 + 159553x^2 + 704392x - 545300 = 0$, (1)

The *zhi fen* method is the same as the modern method of transforming an equation into another whose root is m times the root of the original equation. Thus in solving this problem we multiply the second term of expression (1) by 576, the third term by squared 576, the fourth term by cubed 576, and so on. Dividing this expression by 576 in order to simplify the problem we obtain the expression:

$y^4 + 15792y^3 + 91902528y^2 + 233700360192y - 104208452812800 = 0$.

Solving this expression we obtain the root 384 which is 576 times the required root. Therefore $\frac{384}{576}$ or $\frac{2}{3}$ is the fractional part of the required root. (C)

四元玉鉴　卷中

如意混和　二问

1.

【原文】

今有金球、银球、玉球各一只，共积三十二寸五万五千二百六十四分寸之一万一千三十一，计重一秤一十斤一十一两一十八铢一万三千八百一十六分铢之一万三千六百一十一。[1] 只云：金圆周多如银圆周一寸，银圆周却多玉圆周一寸。金圆周依古法，银圆周依徽术，玉圆周从密率。金方一寸重一十五两一十八铢，银方一寸重一十二两六铢，金、银方寸之重皆按《张丘建》术。玉方一寸重七两。按《黄帝九章》法。[2] 问：三圆周及积寸、重各几何？

答曰：金圆周九寸，积一十五寸一十六分寸之三，重一十四斤一十五两四铢八分铢之七；

银圆周八寸，积一十寸一百五十七分寸之三十，重七斤一十二两二十铢一百五十七分铢之二十八；

玉圆周七寸，积六寸三百五十二分寸之二百八十九，重二斤一十五两一十七铢四十四分铢之四十一。

术曰：立天元一为金圆周，如积求之。得五百三十六万八千一百一十三为益实，四万九千四百六十四为从方，二万九千六百八十二为益廉，一万五十一为从隅，立方开之，[3] 得金圆周。　又：立天元一为银圆周，如积求之。得五百三十三万八千二百八十为益实，二万二百五十三为从方，四百七十一为从廉，一万五十一为从隅，立方开之，[4] 得银圆周。　又：立天元一为玉圆周，如积求之。得五百三十万七千五百五为益实，五万一千三百四十八为从方，三万六百二十四为从廉，一万五十一为从隅，立方开之，[5] 得玉圆周。合问。

BOOK Ⅱ
Ru Yi Hun He (Mixed as You Please [or Various Figures])
2 Problems

1. The sum of the volumes of three balls, one of gold, one of silver, and the other of jade, is $32\frac{11031}{55264}$ (cubic) inches. Their combined weight is 1 *cheng* 10 *jin* 11 *liang* 18 $\frac{13611}{13816}$ *zhu*. [1] It is said that the circumference of the gold ball exceeds the circumference of the silver by one inch, and the circumference of the silver exceeds the circumference of the jade by one inch. It is required that we use the ancient value (of π) for the gold ball, Hui' s value for the silver, and the *mi* value for the jade. According to *Zhang Qiujian*, a cubic inch of gold weighs 15 *liang* 18 *zhu*, and a cubic inch of silver 12 *liang* 6 *zhu*. According to *The Nine Chapters of Huang Di*, a cubic inch of jade weighs 7 *liang*. [2] Find the circumferences, volumes, and weights of the three balls.

Ans. Gold ball:

circumference, 9 inches;

volume, $15\frac{3}{16}$ (cubic) inches;

weight, 14 *jin* 15 *liang* $4\frac{7}{8}$ *zhu*.

Silver ball:

circumference, 8 inches;

volume, $10\frac{30}{157}$ (cubic) inches;

weight, 7 *jin* 12 *liang* $20\frac{28}{157}$ *zhu*.

Jade ball:

【注释】

[1] 记金、银、玉球的体积分别是 V_1, V_2, V_3, 此即: $V_1+V_2+V_3=32\frac{11031}{55264}$ 寸。记金、银、玉球的重量分别是 g_1, g_2, g_3, 此即: $g_1+g_2+g_3=1$ 秤 10 斤 11 两 $18\frac{13611}{13816}$ 铢。(郭)

[2] 记金、银、玉球的周长分别是 l_1, l_2, l_3, 此即: $l_1-l_2=1$, $l_2-l_3=1$。古法是周 3 径 1, 徽术是 $\frac{157}{50}$, 密率是 $\frac{22}{7}$, 依《九章算术》开立圆术, 球体积 $V=\frac{9}{16}d^3$、其中 d 为球的直径, 则 $V_1=\frac{9}{16}d_1^3=\frac{9}{16}(\frac{1}{3}l_1)^3=\frac{1}{48}l_1^3$, $V_2=\frac{9}{16}d_2^3=\frac{9}{16}(\frac{50}{157}l_2)^3=\frac{140625}{7739786}l_2^3$, $V_3=\frac{9}{16}d_3^3=\frac{9}{16}(\frac{7}{22}l_3)^3=\frac{3087}{170368}l_3^3$。(郭)

[3] 开方式的现代形式为:

$10051x^3-29682x^2+49464x-5368113=0$。(陈)

[4] 开方式的现代形式为:

$10051x^3+471x^2+20253x-5338280=0$。(陈)

[5] 开方式的现代形式为:

$10051x^3+30624x^2+51348x-5307505=0$。(陈)

【今译】

今有金球、银球、玉球各一只, 其体积共为 $32\frac{11031}{55264}$ 寸, 共重 1 秤 10 斤 11 两 $18\frac{13611}{13816}$ 铢。只云: 金圆周比银圆周多 1 寸, 银圆周比玉圆周多 1 寸。金圆周按照古法, 银圆周按照徽率, 玉圆周从密率。金 1 寸^{3}重 15 两 18 铢, 银 1 寸^{3}重 12 两 6 铢, 金、银 1 寸^{3}的重量皆按《张丘建算经》。玉 1 寸^{3}重七两。按照《黄帝九章》的标准。问三种圆周及它们的积寸、重各为多少?

答: 金圆周 9 寸, 体积 $15\frac{3}{16}$ 寸^3, 重 14 斤 15 两 $4\frac{7}{8}$ 铢。

银圆周 8 寸, 体积 $10\frac{30}{157}$ 寸^3, 重 7 斤 12 两 $20\frac{28}{157}$ 铢。

玉圆周 7 寸, 体积 $6\frac{289}{352}$ 寸^3, 重 2 斤 15 两 $17\frac{41}{44}$ 铢。

circumference, 7 inches;

volume, $6\frac{289}{352}$ (cubic) inches;

weight, 2 *jin* 15 *liang* $17\frac{41}{44}$ *zhu*.

Process. Let the element *tian* be the circumference of the gold ball. From the statement we have 5368113 for the negative *shi*, 49464 for the positive *fang*, 29682 for the negative *lian*, and 10051 for the positive *yu*, an expression [3] of the third degree whose root is the circumference of the gold ball. Again let the element *tian* be the circumference of the silver ball. From the statement we have 5338280 for the negative *shi*, 20253 for the positive *fang*, 471 for the positive *lian*, and 10051 for the positive *yu*, an expression [4] of the third degree whose root is the circumference of the silver ball. Again let the element *tian* be the circumference of the jade ball. From the statement we have 5307505 for the negative *shi*, 51348 for the positive *fang*, 30624 for the positive *lian*, and 10051 for the positive *yu*, an expression [5] of the third degree whose root is the circumference of the jade ball.

【Notes】

[1] Let the gold ball's volume be V_1, the silver ball's volume V_2, and the jade ball's volume V_3. That is, $V_1 + V_2 + V_3 = 32\frac{11031}{55264}$ (cubic) inches. Let the gold ball's weight be g_1, the silver ball's weight g_2, and the jade ball's weight g_3. That is, $g_1 + g_2 + g_3 =$ 1 *cheng* 10 *jin* 11 *liang* $18\frac{13611}{13816}$ *zhu*. (G)

[2] Let the circumference of the gold ball be l_1, the circumference of the silver ball l_2, and the circumference of the jade ball l_3. That is, $l_1 - l_2 = 1, l_2 - l_3 = 1$. The ancient value (of π) is 3. Hui's value (of π) is $\frac{157}{50}$, and *mi* value (of π) $\frac{22}{7}$. According to the *kai li yuan* method in *The Nine Chapter of Mathematical Procedures*,

术：设天元一为金圆周，以如积方法求其解。得到 −5368113 为常数项，49464 为一次项系数，−29682 为二次项系数，10051 为最高次项系数，开立方，便得到金圆周。又：设天元一为银圆周，以如积方法求其解。得到 −5338280 为常数项，20253 为一次项系数，471 为二次项系数，10051 为最高次项系数，开立方，便得到银圆周。又：设天元一为玉圆周，以如积方法求其解。得到 −5307505 为常数项，51348 为一次项系数，30624 为二次项系数，10051 为最高次项系数，开立方，便得到玉圆周。符合所问。

2.

【原文】

今有三角垛、四角垛果子、方箭、圆箭、平圆径、立圆径、平方面、立方面、茭草垛各一所，共积一万五百八十九算。[1] 只云：立方面不及三角底面一个，如平方面五分之二。茭草底子多三角底面一束，却与立圆径等。圆箭外周如四角底面太半，如方箭外周中半。三角、四角底面相和得三十三个。平圆径多于四角底面七分之四。[2] 问：九事各几何？

答曰：三角底子一十五个，四角底子一十八个；

方箭外周二十四只，圆箭外周一十二只；

平圆径四十二尺，立圆径一十六尺；

平方面三十五尺，立方面一十四尺；

茭草底子一十六束。

术曰：立天元一为三角底子，如积求之。得二百八十四万六千八百三十五为正实，六十万八千四百三十九为益方，一万八千八百六十五为

the volume of a ball is as follows: $V=\frac{9}{16}d^3$, The d is the diameter of the ball. Therefore, $V_1=\frac{9}{16}d_1^3=\frac{9}{16}(\frac{1}{3}l_1)^3=\frac{1}{48}l_1^3$, $V_2=\frac{9}{16}d_2^3=\frac{9}{16}(\frac{50}{157}l_2)^3=\frac{140625}{7739786}l_2^3$, $V_3=\frac{9}{16}d_3^3=\frac{9}{16}(\frac{7}{22}l_3)^3=\frac{3087}{170368}l_3^3$. (G)

[3] The expression in modern form is the equation: $10051x^3-29682x^2+49464x-5368113=0$. (C)

[4] The expression in modern form is the equation: $10051x^3+471x^2+20253x-5338280=0$. (C)

[5] The expression in modern form is the equation: $10051x^3+30624x^2+51348x-5307505=0$. (C)

2. There are two piles of fruit, one with a triangular base and the other with a square base; two bundles of arrows, one round and the other square; a circle, a sphere, and a pile of hay. The sum of their volumes (and areas) is 10589 suan.[1] It is said that an edge of the cube is less by 1 than a side of the triangular base but is two-fifths of a side of the square base; a side of the base of the pile of hay exceeds a side of the base of the triangular pile of fruit by 1 and equals the diameter of the sphere; the circumference of the round bundle of arrows equals the great-half of a side of the square base of the pile of fruit and equals also the middle-half of the perimeter of the square bundle of arrows; the sum of the sides of the triangular base and the square base is 33; and the diameter of the circle exceeds a side of the square base by four-sevenths (of itself). [2] Find the nine dimensions.

Ans. Side of the triangular base, 15;

side of the square base, 18;

从廉，六百三为从隅，立方开之，[3] 得三角底子。 又：立天元一为四角底子，如积求之。得二千四百九十八万二千三百四十四为正实，二百六十万六千六百五十二为益方，七万八千五百六十二为从廉，六百三为益隅，立方开之，[4] 得四角底子。 又：立天元一为方箭外周，如积求之。得八千八百八十二万六千一百一十二为正实，六百九十五万一千七十二为益方，一十五万七千一百二十四为从廉，九百单四半为益隅，立方开之，[5] 即得方箭外周。 又：立天元一为圆箭外周，如积求之。得二千二百二十万六千五百二十八为正实，三百四十七万五千五百三十六为益方，一十五万七千一百二十四为从廉，一千八百九为益隅，立方开之，[6] 得圆箭外周。 又：立天元一为平圆径，如积求之。得九亿五千二百一十万四千八百八十八为正实，四千二百五十七万五千三百一十六为益方，五十四万九千九百三十四为从廉，一千八百九为益隅，立方开之，[7] 得平圆径。 又：立天元一为立圆径，如积求之。得三百四十七万三千五百三十六为正实，六十四万四千三百六十为益方，一万七千五十六为从廉，六百三为从隅，立方开之，[8] 得立圆径。 又：立天元一为平方面，如积求之。得七百五万五千八百二十五为正实，七十一万一千一百二十五为益方，一万三百三十七为从廉，一百二十尺[9] 六分为从隅，立方开之，得平方面。[10] 又：立天元一为立方面，如积求之。得二百二十五万七千八百六十四为正实，五十六万八千九百为益方，二万六百七十四为从廉，六百三为从隅，立方开之，[11] 得立方面。 又：立天元一为茭草底子，如积求之。得三百四十七万三千五百三十六为正实，六十四万四千三百六十为益方，一万七千五十六为从廉，六百三为从隅，立方开之，[12] 得茭草底子。合问。

perimeter of the square bundle of arrows, 24;

circumference of the round bundle of arrows, 12;

diameter of the circle, 42 *chi*;

diameter of the sphere, 16 *chi*;

side of the square, 35 *chi*;

side of the cube, 14 *chi*;

side of the pile of hay, 16.

Process. Let the element *tian* be a side of the triangular base of the pile of fruit. From the statement we have 2846835 for the positive *shi*, 608439 for the negative *fang*, 18865 for the positive *lian*, and 603 for the positive *yu*, a cubic expression [3] whose root is a side of the triangular base. Again let the element *tian* be a side of the square base of the pile of fruit. From the statement we have 24982344 for the positive *shi*, 2606652 for the negative *fang*, 78562 for the positive *lian*, and 603 for the negative *yu*, a cubic expression[4] whose root is a side of the square base. Again let the element *tian* be the perimeter of the square bundle of arrows. From the statement we have 88826112 for the positive *shi*, 6951072 for the negative *fang*, 157124 for the positive *lian*, and $904\frac{1}{2}$ for the negative *yu*, a cubic expression[5] whose root is the perimeter of the square bundle of arrows. Again let the element *tian* be the circumference of the round bundle of arrows. From the statement we have 22206528 for the positive *shi*, 3475536 for the negative *fang*, 157124 for the positive *lian*, and 1809 for the negative *yu*, a cubic expression [6] whose root is the circumference of the round bundle of arrows. Again let the element *tian* be the diameter of the circle. From

【注释】

[1] 各人算者的单位有个、只、尺、束之异，故以算记其积之总数。记三角垛、四角垛之积分别是 V_1，V_2，方箭、圆箭的只数分别为 u_1，u_2，平圆径、立圆径分别为 d_1，d_2，平方面、立方面分别为 a_1，a_2，茭草底子为 v。此即：$V_1 + V_2 + u_1 + u_2 + d_1 + d_2 + a_1 + a_2 + v = 10589$。（郭）

[2] 记三角底子、四角底子、茭草底子、方箭外周、圆箭外周分别为 c_1，c_2，c_3，l_1，l_2，此即：$c_1 - a_2 = 1$，$a_2 = \frac{2}{5}a_1$，$c_3 - c_1 = 1$，$c_3 = d_2$，$l_2 = \frac{2}{3}c_2$，$l_2 = \frac{1}{2}l_1$，$c_1 + c_2 = 33$，$d_1 - c_2 = \frac{4}{7}d_1$。（郭）

[3] 开方式的现代形式为：

$603x^3 + 18865x^2 - 608439x + 2846835 = 0$。（陈）

[4] 开方式的现代形式为：

$-603x^3 + 78562x^2 - 2606652x + 24982344 = 0$。（陈）

[5] 开方式的现代形式为：

$-904\frac{1}{2}x^3 + 157124x^2 - 6951072x + 88826112 = 0$。（陈）

又，罗士琳删此下之“即”字，无必要。（郭）

[6] 开方式的现代形式为：

$-1809x^3 + 157124x^2 - 3475536x + 22206528 = 0$。（陈）

[7] 开方式的现代形式为：

$-1809x^3 + 549934x^2 - 42575316x + 952104888 = 0$。（陈）

[8] 开方式的现代形式为：

$603x^3 + 17056x^2 - 644360x + 3473536 = 0$。（陈）

[9]“一百二十尺”，原文讹作“一百二十步”，今依算校正。“尺”下“六分”，即 0.6 尺。（郭）

[10] 开方式的现代形式为：

$120.6x^3 + 10337x^2 - 711125x + 7055825 = 0$。（陈）

[11] 开方式的现代形式为：

the statement we have 952104888 for the positive *shi*, 42575316 for the negative *fang*, 549934 for the positive *lian*, and 1809 for the negative *yu*, a cubic expression[7] whose root is the diameter of the circle. Again let the element *tian* be the diameter of the sphere. From the statement we have 3473536 for the positive *shi*, 644360 for the negative *fang*, 17056 for the positive *lian*, and 603 for the positive *yu*, a cubic expression[8] whose root is the diameter of the sphere. Again let the element *tian* be a side of the square. From the statement we have 7055825 for the positive *shi*, 711125 for the negative *fang*, 10337 for the positive *lian*, and 120 *chi*[9] 6 *fen* for the positive *yu*, a cubic expression[10] whose root is a side of the square.

Again let the element *tian* be a side of the cube. From the statement we have 2257864 for the positive *shi*, 568900 for the negative *fang*, 20674 for the positive *lian*, and 603 for the positive *yu*, a cubic expression[11] whose root is a side of the cube. Again let the element *tian* be the base of the pile of hay. From the statement we have 3473536 for the positive *shi*, 644360 for the negative *fang*, 17056 for the positive *lian*, and 603 for the positive *yu*, a cubic expression[12] whose root is the base of the pile of hay.

【Notes】

[1] The units of some dimensions are different. There are *ge*, *zhi*, *chi*, and *shu*. Therefore, we can use *suan* for the sum of the volumes (and areas) of all dimentions. Let the volume of the pile with a triangular base be V_1, the volume of the pile with a square base V_2, the arrows number of the square bundle u_1, the arrows number of the round

$603x^3 + 20674x^2 - 568900x + 2257864 = 0$。（陈）

[12] 开方式的现代形式为：

$603x^3 + 17056x^2 - 644360x + 3473536 = 0$。（陈）

【今译】

今有三角垛、四角垛果子，方箭、圆箭、平圆径、立圆径、平方面、立方面、茭草垛各一所，其积共为10589算。只云：立方面比三角底子少1个，是平方面的$\frac{2}{5}$。茭草底子比三角底子多1束，与立圆径相等。圆箭外周是四角底子的$\frac{2}{3}$，是方箭外周的$\frac{1}{2}$。三角、四角底子之和得33个。平圆径比四角底子多平面径的$\frac{7}{4}$。问三角底子、四角底子、方箭外周、圆箭外周、平圆径、立圆径、平方面、立方面、茭草底子这九种各为多少？

答：三角底子15个，四角底子18个；

方箭外周24只，圆箭外周12只；

平圆径42尺，立圆径16尺；

平方面35尺，立方面14尺；

茭草底子16束。

术：设天元一为三角底子，以如积方法求其解。得到2846835为常数项，−608439为一次项系数，18865为二次项系数，603为最高次项系数，开立方，便得到三角底子。　又：设天元一为四角底子，以如积方法求其解。得到24982344为常数项，−2606652为一次项系数，78562为二次项系数，−603为最高次项系数，开立方，便得到四角底子。又：设天元一为方箭外周，以如积方法求其解。得到88826112为常数项，−6951072为一次项系数，157124为二次项系数，$-904\frac{1}{2}$为最高次项系数，开立方，便得到方箭外周。　又：设天元一为圆箭外周，以如

bundle u_2, the diameter of the circle d_1, the diameter of the sphere d_2, the side of the square a_1, the cube a_2, and the base of the pile of hay v. That is, $V_1 + V_2 + u_1 + u_2 + d_1 + d_2 + a_1 + a_2 + v = 10589$. (G)

[2] Let the triangular base be c_1, the square base c_2, the base of the pile of hay c_3, the perimeter of the square bundle of arrows l_1, and the circumference of the round bundle of arrows l_2. That is, $c_1 - a_2 = 1, a_2 = \frac{2}{5}a_1, c_3 - c_1 = 1, c_3 = d_2, l_2 = \frac{2}{3}c_2, l_2 = \frac{1}{2}l_1$, $c_1 + c_2 = 33, d_1 - c_2 = \frac{4}{7}d_1$. (G)

[3] The expression in modern form is the equation: $603x^3 + 18865x^2 - 608439x + 2846835 = 0$. (C)

[4]The expression in modern form is the equation: $-603x^3 + 78562x^2 - 2606652x + 24982344 = 0$. (C)

[5] The expression in modern form is the equation: $-904\frac{1}{2}x^3 + 157124x^2 - 6951072x + 88826112 = 0$. (C)

And it is unnecessary that Luo Shilin deleted the following character *ji*. (G)

[6] The expression in modern form is the equation: $-1809x^3 + 157124x^2 - 3475536x + 22206528 = 0$. (C)

[7] The expression in modern form is the equation: $-1809x^3 + 549934x^2 - 42575316x + 952104888 = 0$. (C)

[8] The expression in modern form is the equation: $603x^3 + 17056x^2 - 644360x + 3473536 = 0$. (C)

[9] The original text is 120 *bu*. It is a mistake. Now I revise it according to the calculation. The following 6 *fen* is 0.6 *chi*. (G)

[10]The expression in modern form is the equation: $120.6x^3 + 10337x^2 - 711125x$

积方法求其解。得到 22206528 为常数项，−3475536 为一次项系数，157124 为二次项系数，−1809 为最高次项系数，开立方，便得到圆箭外周。 又：设天元一为平圆径，以如积方法求其解。得到952104888为常数项，−42575316 为一次项系数，549934 为二次项系数，−1809为最高次项系数，开立方，便得到平圆径。 又：设天元一为立圆径，以如积方法求其解。得到3473536为常数项，−644360为一次项系数，17056 为二次项系数，603 为最高次项系数，开立方，便得到立圆径。 又：设天元一为平方面，以如积方法求其解。得到7055825 为常数项，−711125 为一次项系数，10337 为二次项系数，$120\frac{6}{10}$为最高次项系数，开立方，便得到平方面。 又：设天元一为立方面，以如积方法求其解。得到2257864为常数项，−568900为一次项系数，20674为二次项系数，603为最高次项系数，开立方，便得到立方面。 又：设天元一为茭草底子，以如积方法求其解。得到3473536为常数项，−644360为一次项系数，17056为二次项系数，603为最高次项系数，开立方，便得到茭草底子。符合所问。

$+ 7055825 = 0$. (C)

[11] The expression in modern form is the equation: $603x^3 + 20674x^2 - 568900x + 2257864 = 0$. (C)

[12] The expression in modern form is the equation: $603x^3 + 17056x^2 - 644360x + 3473536 = 0$. (C)

方圆交错 九问

1.

【原文】

今有方、圆田各一段。圆从古法。二积相乘得一万五千五百五十二步。[1] 只云：方田面除圆田周，得三步。[2] 问：方面、圆周各几何？

答曰：方面一十二步，圆周三十六步。

术曰：立天元一为方田面，如积求之。得二万七百三十六为益实，一为正隅，开三乘方除之，[3] 得方田面一十二步。又：立天元一为圆田周，如积求之。得一百六十七万九千六百一十六为益实，一为正隅，三乘方开之，[4] 得圆田周。合问。

【注释】

[1] 记方田的面积、边长分别为 S_1，a，则 $S_1=a^2$；圆田的面积、周长分别为 S_2，l，圆率从古法周3径1，则 $S_2=\frac{1}{12}l^2$。此即：$S_1S_2=\frac{1}{12}a^2l^2=15552$。（郭）

[2] 此即：$l\div a=3$。（郭）

[3] 开方式的现代形式为：$x^4-20736=0$。（陈）

[4] 开方式的现代形式为：$x^4-1679616=0$。（陈）

【今译】

今有方田、圆田各一块。圆率依从古法。二块面积相乘得15552步。只云：圆田的周长除以方田的边长，得3。问：方田边长、圆田周长各为多少？

答：方田边长12步，圆田周长36步。

术：设天元一为方田的边长，以如积方法求其解。得到 −20736 为常数项，1为最高次项系数，开四次方，得到方田边长12步。又：设天元一为圆田周长，以如积方法求其解。得到 −1679616 为常数项，1为最高次项系数，开四次方，得到圆田周长，符合问题。

Fang Yuan Jiao Cuo (Containing Squares and Circles)

9 Problems

1. The product of the areas of two farms, one in the form of a circle and the other a square, is 15552 *bu*.[1] It is said that a quotient of the circumference of the circle and a side of the square is 3 *bu*.[2] Find the side of the square and the circumference of the circle.

Ans. Side of the square, 12 *bu*;

circumference of the circle, 36 *bu*.

Process. Let the element *tian* be a side of the square farm. From the statement we have 20736 for the negative *shi*, and 1 for the positive *yu*, a biquadratic expression[3] whose root is the required side. Again let the element *tian* be the circumference of the circular farm. From the statement we have 1679616 for the negative *shi*, and 1 for the positive *yu*, a biquadratic expression[4] whose root is the required circumference.

【 Notes 】

[1] Let the area and the side of the square be S_1 and a, then, $S_1 = a^2$. Let the area and the circumference of the circle be S_2 and l. We take the ancient value (of π), namely, 3. Then, $S_2 = \frac{1}{12} l^2$. That is, $S_1 S_2 = \frac{1}{12} a^2 l^2 = 15552$. (G)

[2] That is, $l \div a = 3$. (G)

[3] The expression in modern form is the equation: $x^4 - 20736 = 0$. (C)

[4] The expression in modern form is the equation: $x^4 - 1679616 = 0$. (C)

2.

【原文】

今有方、圆田各一段。圆从徽术。共积二百四十七步一百五十七分步之二十九。[1] 只云：方面自乘，内加圆周，共得一百八十步。[2] 问：圆周、方面各几何？

答曰：圆周三十六步，方面一十二步。

术曰：立天元一为圆田周，如积求之。得二万一千九十六为益实，三百一十四为益方，二十五为正隅，平方开之[3]，得圆田周。　又：立天元一为方田面，如积求之。得七十三万二千三百八十四为正实，八千六百八十六为益上廉，二十五为正隅，三乘方开之，[4] 得方田面。合问。

【注释】

[1] 圆率从徽术：周 157 径 50，则 $S_2=\frac{25}{314}l^2$，此即：

$S_1+S_2=a^2+\frac{25}{314}l^2=247\frac{29}{157}$。（郭）

[2] 此即：$a^2+l=180$。（郭）

[3] 开方式的现代形式为：$25x^2-314x-21096=0$。（陈）

[4] 开方式的现代形式为：$25x^4-8686x^2+732384=0$。（陈）

【今译】

今有方田、圆田各一块。圆率依从徽术。二块面积之和得 $247\frac{29}{157}$ 步。只云：方田边长自乘，加圆田周长，共得 180 步。问：圆田周长、方田边长各为多少？

答：圆田周长 36 步，方田边长 12 步。

术：设天元一为圆田周长，以如积方法求其解。得到 −21096 为常数项，−314 为一次项系数，25 为最高次项系数，开平方，得到圆田周长。　又：设天元一为方田边长，以如积方法求其解。得到 732384 为常数项，−8686 为二次项系数，25 为最高次项系数，开四次方，得到方田边长。符合所问。

2. The sum of the areas of a circular and a square farm is $247\frac{29}{157}$ *bu*.[1] It is said that the area of the square farm increased by the circumference of the circular is 180 *bu*[2]. Find the circumference of the circular farm and a side of the square.

Ans. Circumference of the circular, 36 *bu*;

side of the square, 12 *bu*.

Process. Let the element *tian* be the circumference of the circular farm. From the statement we have 21096 for the negative *shi*, 314 for the negative *fang*, and 25 for the positive *yu*, a quadratic expression[3] whose root is the required circumference. Again let the element *tian* be a side of the square farm. From the statement we have 732384 for the positive *shi*, 8686 for the negative upper *lian* and 25 for the positive *yu*, a biquadratic expression[4] whose root is the required side of the square.

[Notes]

[1] Hui's value (of π) is taken, namely, $\pi = \frac{157}{50}$. Then, $S_2 = \frac{25}{314}l^2$. That is, $S_1 + S_2 = a^2 + \frac{25}{314}l^2 = 247\frac{29}{157}$. (G)

[2] That is, $a^2 + l = 180$. (G)

[3] The expression in modern form is the equation: $25x^2 - 314x - 21096 = 0$. (C)

[4] The expression in modern form is the equation: $25x^4 - 8686x^2 + 732384 = 0$. (C)

3.

【原文】

今有方、圆田各一段。圆从密率。方田积内减圆田周，圆田积内减方田面，余二数并得一百九十九步一十一分步之一。[1] 只云：圆周幂减方面，余一千二百八十四步。[2] 问：方面、圆周各几何？

答曰：圆周三十六步，方面一十二步。

术曰：立天元一为圆田周，如积求之。得一亿四千五百一十七万七千二百为正实，八十八为益方，二十二万六千六十五为益上廉，八十八为正隅，三乘方开之，[3] 得圆周。合问。

【注释】

[1] 圆率从密率：周22径7，则 $S_2=\frac{7}{88}l^2$ 此即：

$(S_1-l)+(S_2-a)=(a^2-l)+(\frac{7}{88}l^2-a)=199\frac{1}{11}$。（郭）

[2] 此即：$l^2-a=1284$。（郭）

[3] 开方式的现代形式为：

$88x^4-226065x^2-88x+145177200=0$。（陈）

【今译】

今有方田、圆田各一块。圆率依从密率。方田面积内减去圆田周长，圆田面积内减去方田边长，二余数之和得 $199\frac{1}{11}$ 步。只云：圆田周长之幂减去方田边长，余1284步。问：方田边长、圆田周长各为多少？

答：圆田周长36步，方田边长12步。

术：设天元一为圆田周长，以如积方法求其解。得到145177200为常数项，−88为一次项系数，−226065为二次项系数，88为最高次项系数，开四次方，得到圆田周长。符合所问。

3. There are two farms, one circular and the other square. From the area of the square farm take the circumference of the circular and add to this difference the area of the circular minus a side of the square. The sum is $199\frac{1}{11}$ *bu*.[1] It is said that subtracting the side of the square farm from the square of the circumference of the circular farm gives 1284 *bu*.[2] Find a side of the square farm and the circumference of the circular, using the *mi* value of π.

Ans. Circumference of the circular farm, 36 *bu*;

side of the square, 12 *bu*.

Process. Let the element *tian* be the circumference of the circular farm. From the statement we have 145177200 for the positive *shi*, 88 for the negative *fang*, 226065 for the negative upper *lian*, and 88 for the positive *yu*, a biquadratic expression[3] whose root is the required circumference.

[Notes]

[1] Mi value (of π) is taken, namely, $\pi = \frac{22}{7}$. Then, $S_2 = \frac{7}{88}l^2$. That is, $(S_1 - l) + (S_2 - a) = (a^2 - l) + (\frac{7}{88}l^2 - a) = 199\frac{1}{11}$. (G)

[2] That is, $l^2 - a = 1284$. (G)

[3] The expression in modern form is the equation: $88x^4 - 226065x^2 - 88x + 145177200 = 0$. (C)

4.

【原文】

今有方、圆田各一段。圆从古法。圆田积加方田面于上，又方田积加圆田周，内减上，余六十步。[1] 只云：圆周、方面相和四十八步[2]。问：圆周、方面各几何？

答曰：圆周三十六步，方面一十二步。

术曰：立天元一为圆田周，如积求之。得二万六千三百五十二为正实，一千一百二十八为益方，一十一为正隅，平方开之，[3] 得圆田周。合问。

【注释】

[1] 此即：$(S_1+l)-(S_2+a)=(a^2+l)-(\frac{1}{12}l^2+a)=60$。（郭）

[2] 此即：$l+a=48$。（郭）

[3] 开方式的现代形式为：

$11x^2-1128x+26352=0$。（陈）

【今译】

今有方田、圆田各一块。圆率依从古法。圆田面积加上方田边长，寄于上；又方田面积加上圆田周长，减去上，余60步。只云：圆田周长与方田边长二者相加为48步。问：圆田周长、方田边长各为多少？

答：圆田周长36步，方田边长12步。

术：设天元一为圆田周长，以如积方法求其解。得到26352为常数项，−1128为一次项系数，11为最高次项系数，开平方，得到圆田周长。符合所问。

5.

【原文】

今有方、圆田各一段。圆从徽率。方田幂内减圆田积，余，以方田幂乘之，得五千八百七十七步一百五十七分步之六十三。[1] 只云：方田面如圆田周三分之一[2]。问：方田面、圆田周各几何？

4. Subtract the sum of the area of a circular piece of land and a side of a square piece of land from the sum of the area of the square piece of land and the circumference of the circular. The remainder is 60 *bu*.[1] It is said that the circumference of the circle added to a side of the square equals 48 *bu*[2]. Find the circumference of the circle and a side of the square using the ancient value of π.

Ans. Circumference of the circle, 36 *bu*;

side of the square, 12 *bu*.

Process. Let the element *tian* be the circumference of the circle. From the statement we have 26352 for the positive *shi*, 1128 for the negative *fang*, and 11 for the positive *yu*, a quadratic expression [3] whose root is the required circumference.

[Notes]

[1] That is, $(S_1 + l) - (S_2 + a) = (a^2 + l) - (\frac{1}{12}l^2 + a) = 60$. (G)

[2] That is, $l + a = 48$. (G)

[3] The expression in modern form is the equation: $11x^2 - 1128x + 26352 = 0$. (C)

5. From the area of a square farm subtract the area of a circular farm and multiply the remainder by the area of the square farm. The result is $5877\frac{63}{157}$ *bu*.[1] It is said that a side of the square is one-third of the circumference of the circle [2]. Find a side of the square and the circumference of the circle, using Hui's value of π.

答曰：方面一十二步，圆周三十六步。

术曰：立天元一为方田面，如积求之。得二亿八千九百七十四万四千一百二十八为益实，一万三千九百七十三为正隅，三乘方开之，[3]得方田面。合问。

【注释】

[1] 此即：$(S_1 - S_2) S_1 = (a^2 - \frac{25}{314} l^2) a^2 = 5877\frac{63}{157}$。（郭）

[2] 此即：$a = \frac{1}{3} l$。（郭）

[3] 开方式的现代形式为：$13973x^4 - 289744128 = 0$。（陈）

【今译】

今有方田、圆田各一块。圆率依从徽术。方田的面积内减去圆田的面积，其余数，以方田的面积乘之，得 $5877\frac{63}{157}$ 步。只云：方田边长等于圆田周长的 $\frac{1}{3}$。问：方田边长、圆田周长各为多少？

答：方田边长 12 步，圆田周长 36 步。

术：设天元一为方田边长，以如积方法求其解。得到 −289744128 为常数项，13973 为最高次项系数，开四次方，得到方田的边长。符合所问。

6.

【原文】

今有方、圆田各一段。圆从密率。方田积内减方田面，圆田积内减圆田周，二余数相乘，得八千八百五十六步。[1]只云：方面不及圆周二十四步[2]。问：方面、圆周各几何？

答曰：圆周三十六步，方面一十二步。

术曰：立天元一为圆田周，如积求之。得八百五十七万二千六百八为益实，五十八万八百为益方，九万三千六百三十二为从上廉，四千七百四十一为益下廉，七十七为从隅，三乘方开之，[3]得圆周。合问。

Ans. Side of the square, 12 *bu*;

circumference of the circle, 36 *bu*.

Process. Let the element *tian* be a side of the square. From the statement we have 289744128 for the negative *shi* and 13973 for the positive *yu*, a biquadratic expression [3] whose root is the required side.

【 Notes 】

[1] That is, $(S_1 - S_2)S_1 = (a^2 - \frac{25}{314}l^2)a^2 = 5877\frac{63}{157}$. (G))

[2] That is, $a = \frac{1}{3}l$. (G)

[3] The expression in modern form is the equation: $13973x^4 - 289744128 = 0$. (C)

6. The product of the difference between the area of a square and its side by the difference between the area of a circle and its circumference is 8856 *bu*.[1] It is said that a side of the square is less than the circumference of the circle by 24 *bu* [2]. Find a side of the square and the circumference of the circle using the mi value of π.

Ans. Circumference of the circle, 36 *bu*;

side of the square, 12 *bu*.

【注释】

[1] 此即：$(S_1-a)(S_2-l)=(a^2-a)(\frac{7}{88}l^2-l)=8856$。(郭)

[2] 此即：$l-a=24$。(郭)

[3] 开方式的现代形式为：

$77x^4-4741x^3+93632x^2-580800x-8572608=0$。(陈)

【今译】

今有方田、圆田各一块。圆率依从密率。方田面积内减去方田边长，圆田面积内减去圆田周长，二余数相乘，得8856步。只云：方田边长比圆田周长少24步。问：方田边长、圆田周长各为多少？

答：圆田周长36步，方田边长12步。

术：设天元一为圆田周长，以如积方法求其解。得到−8572608为常数项，−580800为一次项系数，93632为二次项系数，−4741为三次项系数，77为最高次项系数，开四次方，得到圆田周长。符合所问。

7.

【原文】

今有方、圆田各一段。圆从古法。方田积内减圆田积，余，以圆田径乘之，得四百三十二步。[1] 只云：方田周虚加一算，平方开之，不及圆田径五步。[2] 问：方面、圆周各几何？

答曰：圆周三十六步，方面一十二步。

术曰：立天元一为圆田径，如积求之。得六千九百一十二为益实，五百七十六为从方，四百八十为益上廉，一百三十六为从二廉，二十为益下廉，一为正隅，四乘方开之，[3] 得圆田径。三之，为圆周。合问。

【注释】

[1] 此即：$(S_1-S_2)\times\frac{l}{3}=(a^2-\frac{1}{12}l^2)\times\frac{l}{3}=432$。(郭)

[2] 此即：$\frac{l}{3}-\sqrt{4a+1}=5$。(郭)

[3] 开方式的现代形式为：

$x^5-20x^4+136x^3-480x^2+576x-6912=0$。(陈)

Process. Let the element *tian* be the circumference of the circle. From the statement we have 8572608 for the negative *shi*, 580800 for the negative *fang*, 93632 for the positive upper *lian*, 4741 for the negative lower *lian*, and 77 for the positive *yu*, a biquadratic expression [3] whose root is the required circumference.

【 Notes 】

[1] That is, $(S_1 - a)(S_2 - l) = (a^2 - a)(\frac{7}{88}l^2 - l) = 8856$. (G)

[2] That is, $l - a = 24$. (G)

[3] The expression in modern form is the equation: $77x^4 - 4741x^3 + 93632x^2 - 580800x - 8572608 = 0$. (C)

7. The product of the difference between the area of a square and a circular farm by the diameter of the circular is 432 *bu*.[1] It is said that the square root of the perimeter of the square farm increased by 1 is less than the diameter of the circular by 5 *bu*.[2] Find a side of the square and the circumference of the circle using the ancient value of π.

Ans. Circumference of the circle, 36 *bu*;

side of the square, 12 *bu*.

Process. Let the element *tian* be the diameter of the circular farm. From the statement we have 6912 for the negative *shi*, 576 for the positive *fang*, 480 for the negative upper *lian*, 136 for the positive second *lian*, 20 for the negative lower *lian*, and 1 for the positive *yu*, an expression[3] of the fifth degree whose root is the required diameter. This root multiplied by 3 gives the circumference of the circle.

【今译】

今有方田、圆田各一块。圆率依从古法。方田面积内减去圆田面积，其余数，以圆田的直径乘之，得432步。只云：方田的周长加1，开平方，其根比圆田的直径少5步。问：方田边长、圆田周长各为多少？

答：圆田周长36步，方田边长12步。

术：设天元一为圆田直径，以如积方法求其解。得到−6912为常数项，576为一次项系数，−480为二次项系数，136为三次项系数，−20为四次项系数，1为最高次项系数，开五次方，得到圆田直径。以3乘之，得到圆田周长。符合所问。

8.

【原文】

今有方、圆田各一段。圆从徽术。方田积内减圆田周三分之二，余数于上；圆田积内加方田面二分之一，减上，余一十步一百五十七分步之一百二十八。[1] 只云：并方面、圆周为益实，二为益方，三为从廉，一为从隅，立方开之，得数如方田面弱半。[2] 问：圆周、方面各几何？

答曰：圆周三十六步，方面一十二步。

术曰：立天元一为开方数，如积求之。得一万一百八十八为益实，一千八百八十四为从方，一万四百八十八为从上廉，二千七十二为从二廉，二百二十五为从三廉，四百五十为益下廉，七十五为益隅，五乘方开之，[3] 得开方数三步。四之，即方田面。合问。

【注释】

[1] 此即：$\left(S_1-\frac{2}{3}l\right)-\left(S_2+\frac{1}{2}a\right)=\left(a^2-\frac{2}{3}l\right)-\left(\frac{25}{314}l^2+\frac{1}{2}a\right)=10\frac{128}{157}$。（郭）

[2] 此即：$w=\frac{1}{4}a$。其中w是开方式$x^3+3x^2-2x-(a+l)=0$的根。（郭）

[3] 开方式的现代形式为：

$-75x^6-450x^5+225x^4+2072x^3+10488x^2+1884x-10188=0$。（陈）

【Notes】

[1] That is, $(S_1 - S_2) \times \frac{l}{3} = (a^2 - \frac{1}{12}l^2) \times \frac{l}{3} = 432$. (G)

[2] That is, $\frac{l}{3} - \sqrt{4a+1} = 5$. (G)

[3] The expression in modern form is the equation: $x^5 - 20x^4 + 136x^3 - 480x^2 + 576x - 6912 = 0$. (C)

8. From the difference between the area of a square farm and two-thirds of the circumference of a circular subtract the sum of the area of the circular and one-half of a side of the square. The remainder is $10\frac{128}{157}$ *bu*.[1] It is said that by taking the sum of a side of the square farm and the circumference of the circular for the negative *shi*, 2 for the negative *fang*, 3 for the positive *lian*, and 1 for the positive *yu*, we have a cubic expression whose root is equal to the weak-half of a side of the square.[2] Find a side of the square and the circumference of the circle using Hui's value (of π).

Ans. Circumference of the circle, 36 *bu*;

side of the square, 12 *bu*.

Process. Let the element *tian* be the root of the given expression. From the statement we have 10188 for the negative *shi*, 1884 for the positive *fang*, 10488 for the positive upper *lian*, 2072 for the positive second *lian*, 225 for the positive third *lian*, 450 for the negative lower *lian*, and 75 for the negative *yu*, an expression[3] of the sixth degree whose root is 3. This root multiplied by 4 gives a side of the square.

【今译】

今有方田、圆田各一块。圆率依从徽术。方田的面积内减去圆田周长的$\frac{2}{3}$，其余数寄于上；圆田的面积加上方田边长的$\frac{1}{2}$，去减上，余$10\frac{128}{157}$步。只云：方田边长与圆田周长相加作为负实，−2作为一次项系数，3作为二次项系数，1作为最高次项系数，开立方，其根等于方田边长的$\frac{1}{4}$。问：圆田周长、方田边长各为多少？

答：圆田周长36步，方田边长12步。

术：设天元一为开方数，以如积方法求其解。得到−10188为常数项，1884为一次项系数，10488为二次项系数，2072为三次项系数，225为四次项系数，−450为五次项系数，−75为最高次项系数，开六次方，得到开方数三步。以四乘之，就是方田边长。符合所问。

9.

【原文】

今有方、圆田各一段。圆从密率。圆田积内加二个圆田周，减一段方田积，余数于上；又方田积内加三个方田面，减一段圆田积，余数加上；以方田面少半乘之，又以圆田周六分之一乘之，得二千三百四步。[1] 只云：方田面为益实，四为益方，三为从廉，一为正隅，立方开之，得数，以十八乘之，与圆田周等。[2] 问：方面、圆周各几何？

答曰：方面一十二步，圆周三十六步。

术曰：立天元一为开方数，如积求之。得一千一百五十二为益实，五十六为益二廉，三十为从三廉，一十九为从四廉，六为从五廉，一为从隅，六乘方开之，[3] 得二步，为开方数。合问。

【注释】

[1] 此即：

[Notes]

[1] That is, $(S_1 - \frac{2}{3}l) - (S_2 + \frac{1}{2}a) = (a^2 - \frac{2}{3}l) - (\frac{25}{314}l^2 + \frac{1}{2}a) = 10\frac{128}{157}$. (G)

[2] That is, $w = \frac{1}{4}a$. w is a root of the equation $x^3 + 3x^2 - 2x - (a + l) = 0$. (G)

[3] The expression in modern form is the equation: $-75x^6 - 450x^5 + 225x^4 + 2072x^3 + 10488x^2 + 1884x - 10188 = 0$. (C)

9. Add to the area of a circular farm twice its circumference, then subtract the area of a square farm. Set the result thus obtained aside. Add to the area of the square farm 3 times its side and subtract the area of the circular. To this result add the result set aside and multiply by the small-half of a side of the square farm, then by one-sixth of the circumference of the circular. The product thus obtained is 2304 *bu*.[1] It is said that if we put a side of the square farm for the negative *shi*, 4 for the negative *fang*, 3 for the positive *lian*, and 1 for the positive *yu*, the root of this cubic expression multiplied by 18 will equal the circumference of the circular farm.[2] Find a side of the square and the circumference of the circle using *mi* value (of π).

Ans. Side of the square, 12 *bu*;

circumference of the circle, 36 *bu*.

Process. Let the element *tian* be the root of the given expression. From the

$[(S_2+2l-S_1)+(S_1+3a-S_2)]\times\frac{1}{3}a\times\frac{1}{6}l=[(\frac{7}{88}l^2+2l-a^2)+(a^2+3a-\frac{7}{88}l^2)]\times\frac{1}{3}a\times\frac{1}{6}l=2304$。(郭)

[2] 此即：$18w=l$。其中 w 是开方式 $w^3+3w^2-4w-a=0$ 的根。(郭)

[3] 开方式的现代形式为：

$x^7+6x^6+19x^5+30x^4-56x^3-1152=0$。(陈)

【今译】

今有方田、圆田各一块。圆率依从密率。圆田面积加圆田周的2倍，减方田面积，其余数寄于上；又方田面积加方田边长的3倍，减圆田面积，其余数加寄于上者，以方田边长的$\frac{1}{3}$乘之，又以圆田周长的$\frac{1}{6}$乘之，得2304步。只云：方田边长作为负实，−4作为一次项系数，3作为二次项系数，1作为最高次项系数，开立方，以18乘其根，等于圆田周长。问：方田边长、圆田周长各为多少？

答：方田边长12步，圆田周长36步。

术：设天元一为开方数，以如积方法求其解。得到−1152为常数项，−56为三次项系数，30为四次项系数，19为五次项系数，6为六次项系数，1为最高次项系数，开七次方，得到2步，为开方数。符合所问。

statement we have 1152 for the negative *shi*, 56 for the negative second *lian*, 30 for the positive third *lian*, 19 for the positive fourth *lian*, 6 for the positive fifth *lian*, and 1 for the positive *yu*, an expression [3] of the seventh degree whose root, 2 *bu*, is the required root.

[Notes]

[1] That is, $[(S_2 + 2l - S_1) + (S_1 + 3a - S_2)] \times \frac{1}{3}a \times \frac{1}{6}l = [(\frac{7}{88}l^2 + 2l - a^2) + (a^2 + 3a - \frac{7}{88}l^2)] \times \frac{1}{3}a \times \frac{1}{6}l = 2304$. (G)

[2] That is, $18w = l$. w is a root of the equation $w^3 + 3w^2 - 4w - a = 0$. (G)

[3] The expression in modern form is the equation: $x^7 + 6x^6 + 19x^5 + 30x^4 - 56x^3 - 1152 = 0$. (C)

三率究圆 一十四问

1.

【原文】

今有平圆积四十九步三百一十四分步之二百三十九[1]。问：为徽圆周几何？

答曰：二十五步。

术曰：立天元一为徽圆周，如积求之。得一万五千六百二十五为益实，二十五为从隅，平方开之。[2] 合问。

【注释】

[1] 以刘徽的圆周率 $\pi=\frac{157}{50}$ 入算，此即：$S=\frac{25}{314}l^2=49\frac{239}{314}$。（郭）

[2] 开方式的现代形式为：$25x^2-15625=0$。（陈）

【今译】

今有圆田的面积 $49\frac{239}{314}$ 步。问：以徽率计算，圆周为多少？

答：25 步。

术：设天元一为以徽率计算的圆周，以如积方法求其解。得到 -15625 为常数项，25 为最高次项系数，开平方，得到圆周。符合所问。

2.

【原文】

今有平圆积四十九步三百一十四分步之二百三十九。问：为徽圆径几何？

答曰：七步一百五十七分步之一百五十一。

术曰：立天元一为徽圆径，如积求之。得一百五十六万二千五百为益实，二万四千六百四十九为从隅，平方开之。[1] 不尽，以连枝同体术求之。合问。

San Lü Jiu Yuan (Reckoning Circles with the "Three Values of π")
14 Problems

1. The area of a circle is $49\frac{239}{314}$ (square) *bu* [1]. Find the circumference using Hui's value of π.

Ans. 25 *bu*.

Process. Let the element *tian* be the circumference. From the statement we have 15625 for the negative *shi*, and 25 for the positive *yu*, a quadratic expression [2] whose root is the required circumference.

[Notes]

[1] Use Liu Hui's value of π to calculate, namely $\pi = \frac{157}{50}$. That is, $S = \frac{25}{314} l^2 = 49\frac{239}{314}$. (G)

[2] The expression in modern form is the equation: $25x^2 - 15625 = 0$. (C)

2. The area of a circle is $49\frac{239}{314}$ (square) *bu*. Find the diameter using Hui's value of π.

Ans. $7\frac{151}{157}$ *bu*.

Process. Let the element *tian* be the required diameter. From the statement we have 1562500 for the negative *shi*, and 24649 for the positive *yu*, a quadratic expression [1] whose root, a mixed number, is the required diameter. Obtain the integral part of the root by the usual method. For

【注释】

[1] 开方式的现代形式为：$24649x^2-1562500=0$。(陈)

【今译】

今有圆田的面积 $49\frac{239}{314}$ 步。问：以徽率计算，圆径为多少？

答：$7\frac{151}{157}$ 步。

术：设天元一为以徽率计算的圆径，以如积方法求其解。得到 −1562500 为常数项，24649 为最高次项系数，开平方。开方不尽，按照连枝同体术求其分数部分。便符合所问。

3.

【原文】

今有平圆积四十五步一十一分步之九[1]。问：为密圆周几何？

答曰：二十四步。

术曰：立天元一为密圆周，如积求之。得五百七十六为益实，一为正隅，平方开之。[2] 合问。

【注释】

[1] 以圆周率 $\pi=\frac{22}{7}$ 入算，此即：$S=\frac{7}{88}l^2=45\frac{9}{11}$。(郭)

[2] 开方式的现代形式为：$x^2-576=0$。(陈)

【今译】

今有圆田的面积 $45\frac{9}{11}$ 步。问：以密率计算，圆周为多少？

答：24 步。

术：设天元一为以密率计算的圆周，以如积方法求其解。得到 −576 为常数项，1 为最高次项系数，开平方，得到圆周。符合所问。

obtaining the fractional part apply the *lian zhi tong ti* method.

【 Notes 】

[1] The expression in modern form is the equation: $24649x^2 - 1562500 = 0$. (C)

3. The area of a circle is $45\frac{9}{11}$ (square) *bu* [1]. Find the circumference using the mi value of π.

Ans. 24 *bu*.

Process. Let the element *tian* be the circumference. From the statement we have 576 for the negative *shi*, and 1 for the positive *yu*, a quadratic expression [2] whose root is the required circumference.

【 Notes 】

[1] Use $\pi = \frac{22}{7}$ for the calculation, then $S = \frac{7}{88} l^2 = 45\frac{9}{11}$. (G)

[2] The expression in modern form is the equation: $x^2 - 576 = 0$. (C)

4.

【原文】

今有平圆积四十五步一十一分步之七。问：为密圆径几何？

答曰：七步一十一分步之七。

术曰：立天元一为密圆径，如积求之。得七千五十六为益实，一百二十一为从隅，平方开之，[1] 得七步。不尽，按之分法求之。合问。

【注释】

[1] 开方式的现代形式为：$121x^2 - 7056 = 0$。（陈）

【今译】

今有圆田的面积 $45\frac{7}{11}$ 步。问：以密率计算，圆径为多少？

答：$7\frac{7}{11}$ 步。

术：设天元一为以密率计算的圆径，以如积方法求其解。得到 −7056 为常数项，121 为最高次项系数，开平方，得到 7 步。开方不尽，按照之分法求其分数部分。符合所问。

5.

【原文】

今有立圆积九百七十二尺[1]。问：为古立圆径几何？

答曰：一丈二尺。

术曰：立天元一为古立圆径，如积求之。得一万五千五百五十二为益实，九为从隅，立方开之，得一丈二尺。[2] 合问。

【注释】

[1] 记立圆的体积、直径分别为 V, d, 取 $\pi = 3$,《九章算术》的开立圆术给出 $V = \frac{9}{16}d^3$。（郭）

4. The area of a circle is $45\frac{7}{11}$ (square) *bu*. Find the diameter using the *mi* value of π.

Ans. $7\frac{7}{11}$ *bu*.

Process. Let the element *tian* be the diameter. From the statement we have 7056 for the negative *shi*, and 121 for the positive *yu*, a quadratic expression [1] whose root, a mixed number, is the required diameter. Obtain the integral part of the root by the usual method. For obtaining the fractional part apply the *zhi fen* method.

[Notes]

[1] The expression in modern form is the equation: $121x^2 - 7056 = 0$. (C)

5. The volume of a sphere is 972 (cubic) *chi*[1]. Find the diameter using the ancient value of π.

Ans. 1 *zhang* 2 *chi*.

Process. Let the element *tian* be the diameter of the given sphere. From the statement we have 15552 for the negative *shi*, and 9 for the positive *yu*, a cubic expression [2] whose root, 1 *zhang* 2 *chi*, is the required diameter.

[Notes]

[1] Let the volume of the sphere be V, and the diameter d, and use 3 for π. Ac-

[2] 开方式的现代形式为：$9x^2-15552=0$。（陈）

【今译】

今有球的体积972尺。问：以古法计算，立圆的直径为多少？

答：1丈2尺。

术：设天元一为以古法计算的立圆的直径，以如积方法求其解。得到−15552为常数项，9为最高次项系数，开立方，得到1丈2尺。符合所问。

6.

【原文】

今有立圆积九百七十二尺[1]。问：为古立圆周几何？

答曰：三丈六尺。

术曰：立天元一为古立圆周，如积求之。得四万六千六百五十六为益实，一为正隅，立方开之，得三丈六尺。[2] 合问。

【注释】

[1] 记立圆的周长为l，取$\pi=3$，由《九章算术》的开立圆术导出：$V=\frac{9}{16}d^3=\frac{1}{48}l^3$。（郭）

[2] 开方式的现代形式为：$x^3-46656=0$。（陈）

【今译】

今有球的体积972尺。问：以古法计算，立圆的周长为多少？

答：3丈6尺。

术：设天元一为以古法计算的立圆周，以如积方法求其解。得到−46656为常数项，1为最高次项系数，开立方，得到3丈6尺。符合所问。

cording to the *li yuan* method in *The Nine Chapters of Mathematical Procedures*, $V = \frac{9}{16} d^3$. (G)

[2] The expression in modern form is the equation: $9x^2 - 15552 = 0$. (C)

6. The volume of a sphere is 972 (cubic) *chi* [1]. Find the circumference using the ancient value of π.

Ans. 3 *zhang* 6 *chi*.

Process. Let the element *tian* be the circumference of the given sphere. From the statement we have 46656 for the negative *shi*, and 1 for the positive *yu*, a cubic expression [2] whose root, 3 *zhang* 6 *chi*, is the required circumference.

【 Notes 】

[1]Let the circumference of the sphere be l, and use 3 for π. According to the *kai li yuan* method in *The Nine Chapters of Mathematical Procedures*, $V = \frac{9}{16} d^3 = \frac{1}{48} l^3$. (G)

[2] The expression in modern form is the equation: $x^3 - 46656 = 0$. (C)

7.

【原文】

今有立圆积九百二十八尺一百五十七分尺之一百四[1]。问：为徽立圆径几何？

答曰：一丈二尺。

术曰：立天元一为徽立圆径，如积求之。得一千七百二十八为益实，一为正隅，立方开之。[2] 合问。

【注释】

[1] 由题意知，朱世杰认为刘徽的球体积公式是：$V=\frac{1350}{2512}d^3$，不知何据。现传《九章算术注》中刘徽未能完全解决球体积问题，而是"以俟能言者"。（郭）

[2] 开方式的现代形式为：$x^3-1728=0$。（陈）

【今译】

今有球的体积 $928\frac{104}{157}$ 尺。问：以徽术计算，球的直径为多少？

答：1 丈 2 尺。

术：设天元一为以徽术计算的球的直径，以如积方法求其解。得到 -1728 为常数项，1 为最高次项系数，开立方。符合所问。

8.

【原文】

今有立圆积九百二十八尺一百五十七分尺之一百四。问：为徽立圆周几何？

答曰：三丈六尺。

术曰：立天元一为徽立圆周，如积求之。得四万六千六百五十六为益实，一为正隅，立方开之，[1] 得三丈六尺。合问。

7. The volume of a sphere is $928\frac{104}{157}$ (cubic) *chi*[1]. Find the diameter using Hui's value of π.

Ans. 1 *zhang* 2 *chi*.

Process. Let the element *tian* be the diameter of the given sphere. From the statement we have 1728 for the negative *shi*, and 1 for the positive *yu*, a cubic expression [2] whose root is the required diameter.

【Notes】

[1] From the statement we know that Zhu Shijie considered Liu Hui's formula of the volume of a sphere $V=\frac{1350}{2512}d^3$. We do not know why he considered so. According to the existent *Commentary of "The Nine Chapters of Mathematical Procedures"*, Liu did not completely solve the problem of the volume of a sphere. And Liu pointed out that they must wait for the person who could solve it (*yi si neng yan zhe*). (G)

[2] The expression in modern form is the equation: $x^3 - 1728 = 0$. (C)

8. The volume of a sphere is $928\frac{104}{157}$ (cubic) *chi*. Find the circumference using Hui's value (of π).

Ans. 3 *zhang* 6 *chi*.

Process. Let the element *tian* be the circumference of the given sphere. From the statement we have 46656 for the negative *shi*, and 1 for the positive *yu*, a cubic expression [1] whose root is the required diameter.

【注释】

[1] 开方式的现代形式为：$x^3-46656=0$。(陈)

【今译】

今有球的体积 $928\frac{104}{157}$ 尺。问：以徽术计算，球的周长为多少？

答：3 丈 6 尺。

术：设天元一为以徽术计算的球的周长，以如积方法求其解。得到 −46656 为常数项，1 为最高次项系数，开立方，得 3 丈 6 尺。符合所问。

9.

【原文】

今有立圆积九百二十七尺一十一分尺之九[1]。问：为密立圆径几何？

答曰：一丈二尺。

术曰：立天元一为密立圆径，如积求之。得三十二万六千五百九十二为益实，一百八十九为从隅，立方开之。[2] 合问。

【注释】

[1] 现传《九章算术注释》中李淳风等依 $\pi=\frac{22}{7}$ 得出的球体积公式是 $V=\frac{11}{21}d^3$。此问计算有误。术文中的从隅 $189=9\times 21$，是公式的分母与立圆积的分子之积。(郭)

[2] 开方式的现代形式为：$189x^3-326592=0$。(陈)

【今译】

今有球的体积 $927\frac{9}{11}$ 尺。问：以密率计算，球的直径为多少？

答：1 丈 2 尺。

术：设天元一为以密率计算的球的直径，以如积方法求其解。得到 −326592 为常数项，189 为最高次项系数，开立方。符合所问。

[Notes]

[1] The expression in modern form is the equation: $x^3 - 46656 = 0$. (C)

9. The volume of a sphere is $927\frac{9}{11}$ (cubic) *chi*[1]. Find the diameter using the *mi* value of π.

Ans. 1 *zhang* 2 *chi*.

Process. Let the element *tian* be the diameter of the given sphere. From the statement we have 326592 for the negative *shi*, and 189 for the positive *yu*, a cubic expression[2] whose root is the required diameter.

[Notes]

[1] According to the extant Li Chunfeng's *Commentary of "The Nine Chapters of Mathematical Procedures"*, the formula of a sphere given by Li Chunfeng is $V = \frac{11}{21}d^3$ by using $\pi = \frac{22}{7}$. There is a mistake in the problem. The positive *yu* $189 = 9 \times 21$ is the product of the denominator of the formula and the numerator of the volume of the sphere. (G)

[2] The expression in modern form is the equation: $189x^3 - 326592 = 0$. (C)

10.

【原文】

今有立圆积九百二十七尺一十一分尺之九。问：为密立圆周几何？

答曰：三丈六尺。

术曰：立天元一为密立圆周，如积求之。得四万六千六百五十六为益实，一为正隅，立方开之，[1] 得三丈六尺。合问。

【注释】

[1] 开方式的现代形式为：$x^3 - 46656 = 0$。（陈）

【今译】

今有球的体积 $927\frac{9}{11}$ 尺。问：以密率计算，球的周长为多少？

答：3丈6尺。

术：设天元一为以密率计算的球的周长，以如积方法求其解。得到 -46656 为常数项，1为最高次项系数，开立方，得到3丈6尺。符合所问。

11.

【原文】

今有平幂二百六十五尺[1]。问：为平方面几何？

答曰：一十六尺一十一分尺之三。

术曰：立天元一为平方面，如积求之。得二百六十五为益实，一为正隅，平方开之。[2] 不尽，命分。[3] 合问。

【注释】

[1] 此平幂指平方。即：$a^2 = 265$。（郭）

[2] 开方式的现代形式为：$x^2 - 265 = 0$。（陈）

[3] 求出整数部分16后，其减根开方式为 $x^2 - 32x - 9 = 0$。据刘徽《九章算

10. The volume of a sphere is $927\frac{9}{11}$ (cubic) *chi*. Find the circumference using the *mi* value of π.

Ans. 3 *zhang* 6 *chi*.

Process. Let the element *tian* be the diameter of the given sphere. From the statement we have 46656 for the negative *shi*, and 1 for the positive *yu*, a cubic expression [1] whose root is the required circumference.

【 Notes 】

[1] The expression in modern form is the equation: $x^3 - 46656 = 0$. (C)

11. Find a side of the square whose area is 26 (square) *chi* [1].

Ans. $16\frac{3}{11}$ *chi*.

Process. Let the element *tian* be a side of the square. From the statement we have 265 for the negative *shi*, and 1 for the positive *yu*, a quadratic expression [2] whose root, a mixed number, is the required side [3].

【 Notes 】

[1] The *ping mi* refers to the square of a side. That is, $a^2 = 265$. (G)

[2] The expression in modern form is the equation: $x^2 - 265 = 0$. (C)

术注》记载，其分数部分“以借算加定法而命分”求得：$\frac{9}{32+1}=\frac{9}{33}=\frac{3}{11}$。（郭）

此处给出的仅为近似值。其以如下方式求得：

$$\begin{array}{ccc} 1 & 0 & -265\,(\,16 \\ & \underline{16} & \underline{256} \\ & 16 & 9 \\ & \underline{16} & \\ & 32 & \\ & \underline{1} & \\ & 33 & \end{array}$$

因此，$16\frac{9}{33}$或者$16\frac{3}{11}$为近似值。（陈）

【今译】

今有平面面积265尺。问：将其变成正方形，其边长为多少？

答：$16\frac{3}{11}$尺。

术：设天元一为正方形的边长，以如积方法求其解。得到-265为常数项，1为最高次项系数，开平方。开方不尽，以法和余实命名分数部分。符合所问。

12.

【原文】

今有平方面一十六尺一十一分尺之三。问：为平幂几何？

答曰：二百六十五尺。

术曰：立天元一为平幂，如积求之。得三万二千六十五为益实，一百二十一为从方，上实，下法而一。[1] 合问[2]。

【注释】

[1] 开方式的现代形式为：$121x-32065=0$。（陈）

[3] After extracting the integral part of the root 16, the *jian gen* equation is $x^2 - 32x - 9 = 0$. According to Liu Hui's *Commentary of "The Nine Chapters of Mathematical Procedures"*, the fractional part can be solved by *yi jie suan jia ding fa er ming fen*, that is, $\frac{9}{32+1} = \frac{9}{33} = \frac{3}{11}$. (G) The result given here is only the approximate value. It is found in the following manner:

1	0	−265 (16
	16	256
	16	9
	16	
	32	
	1	
	33	

Therefore $16\frac{9}{33}$ or $16\frac{3}{11}$ is the approximate root. (C)

12. A side of a square is $16\frac{3}{11}$ *chi*. Find the area.

Ans. 265 *chi*.

Process. Let the element *tian* be the area of the given square. From the statement we have 32065 for the negative *shi*, and 121 for the positive *fang*, a linear expression [1] whose root is the required area [2].

[Notes]

[1] The expression in modern form is the equation: $121x - 32065 = 0$. (C)

［2］此处给出的结果仅为近似值。其以如下方式求得：

$16 \times 11 = 176$，$(176)^2 = 30976$，$\frac{2 \times 16 + 1}{11} = 3$，

$3 \times 3 \times 121 = 1089$，$\frac{1089 + 30976}{121} = 265$。（陈）

【今译】

今有正方形的一边长为$16\frac{3}{11}$尺。问：其正方形的面积为多少？

答：265尺。

术：设天元一为正方形的面积，以如积方法求其解。得到−32065，作为被除数，121作为除数，除之。符合所问。

13.

【原文】

今有立幂五百七十四尺。问：为立方面几何？

答曰：八尺七分尺之二。

术曰：立天元一为立方面，如积求之。得五百七十四为益实，一为正隅，立方开之。[1]不尽，命分。合问[2]。

【注释】

［1］开方式的现代形式为：$x^3 - 574 = 0$。（陈）

［2］此处给出的结果仅为近似值。其以如下方式求得：

1	0	0	−574（8
	8	64	512
	8	64	62
	8	128	
	16	192	
	8	24	
	24	1	
		217	

因此，$8\frac{62}{217}$或者$8\frac{2}{7}$为所求得近似根。（陈）

[2] The result given here is only the approximate value. It is found in the following manner:

$$16 \times 11 = 176,\ (176)^2 = 30976,\ \frac{2 \times 16 + 1}{11} = 3$$

$$3 \times 3 \times 121 = 1089,\ \frac{1089 + 30976}{121} = 265.\ (C)$$

13. The volume of a cube is 574 (cubic) *chi*. Find an edge.

Ans. $8\frac{2}{7}$ *chi*.

Process. Let the element *tian* be an edge of the given cube. From the statement we have 574 for the negative *shi*, and 1 for the positive *fang*, a cubic expression [1] whose root, a mixed number, is the required edge [2].

[Notes]

[1] The expression in modern form is the equation: $x^3 - 574 = 0$. (C)

[2] The result given is only the approximate value. It is found in the following manner:

1	0	0	−574 (8
	8	64	512
	8	64	62
	8	128	
	16	192	
	8	24	
	24	1	
		217	

Therefore, $8\frac{62}{217}$ or $8\frac{2}{7}$ is the approximate root. (C)

【今译】

今有立体体积 574 尺。问：将其变成正方体，其边长为多少？

答：$8\frac{2}{7}$ 尺。

术：设天元一为正方体的边长，以如积方法求其解。得到 −574 为常数项，1 为最高次项系数，开立方。开方不尽，以法和余实命名分数部分。符合所问。

14.

【原文】

今有立方面八尺七分尺之二。问：为立幂几何？

答曰：五百七十四尺。

术曰：立天元一为立幂，如积求之。得一十九万六千八百八十二为正实，三百四十三为益方，无隅，平方开之。[1] 合问[2]。

【注释】

[1] 开方式的现代形式为：$-343x+196882=0$。（陈）

[2] 此处给出的结果仅为近似值。其求得得方式如下：

$7^3=343$，为方。

$(8\times7)^3=175616$，$8\times3=24$，$24\times8=192$，

$192+24+1=217$，$\frac{217}{7}=31$，$2\times31=62$，$62\times343=21266$，

$21266+175616=196882$，为实。

$\frac{196882}{343}=574$。（陈）

【今译】

今有正方体的边长为 $8\frac{2}{7}$ 尺。问：正方体的体积为多少？

答：574 尺。

术：设天元一为正方体的体积，以如积方法求其解。得到 196882 为常数项，−343 为一次项系数，没有最高次项系数，开平方。符合所问。

14. The side of a cube is $8\frac{2}{7}$ *chi*. Find the volume.

Ans. 574 (cubic) *chi*.

Process. Let the element *tian* be the volume of the given cube. From the statement we have 196882 for the positive *shi*, and 343 for the negative *fang*, a linear expression [1] whose root is the required volume [2].

[Notes]

[1] The expression in modern form is the equation: $-343x + 196882 = 0$. (C)

[2] The result given is only the approximate value. It is found in the following manner:

$7^3 = 343$, for the *fang*.

$(8 \times 7)^3 = 175616$, $8 \times 3 = 24$, $24 \times 8 = 192$,

$192 + 24 + 1 = 217$, $\frac{217}{7} = 31$, $2 \times 31 = 62$, $62 \times 343 = 21266$,

$21266 + 175616 = 196882$, for the shi.

$\frac{196882}{343} = 574$. (C)

明积演段 二十问

1.

【原文】

今有直积一十二步[1]。只云勾弦和八步[2]。问：勾股较几何？

答曰：一步。

术曰：立天元一为较，如积求之，得二十三为正实，二十八为益方，四为从廉，一为从隅，立方开之。[3] 合问。

【注释】

[1] 记直积的广、长分别为 a，b，此即：$ab = 12$。（郭）

[2] 将直积的广、长分别看作勾股形的勾、股，则直积的对角线就是弦，此即 $a + c = 8$。（郭）

[3] 开方式的现代形式为：$x^3 + 4x^2 - 28x + 23 = 0$。（陈）

【今译】

今有长方形的面积 12 步。只云勾弦和为 8 步。问：勾股较为多少？

答：1 步。

术：设天元一为勾股较，以如积方法求其解。得到 23 作为常数项，-28 作为一次项系数，4 作为二次项系数，1 作为最高次项系数，开立方。符合所问。

2.

【原文】

今有直积一十二步。只云勾弦和八步。问：勾弦较几何？

答曰：二步。

术曰：立天元一为勾弦较，如积求之，得七十二为益实，六十四为从方，一十六为益廉，一为正隅，立方开之。[1] 合问。

Ming Ji Yan Duan (Problems on Areas)

20 Problems

1. The *zhi ji* is 12 (square) *bu* [1] , and the sum of the *gou* and the *xian* 8 *bu* [2]. Find the *gu* less the *gou*.

Ans. 1 *bu*.

Process. Let the element *tian* be the *jiao*. From the statement we have 23 for the positive *shi*, 28 for the negative *fang*, 4 for the positive *lian*, and 1 for the positive *yu*, a cubic expression [3] whose root is the required *jiao*.

[Notes]

[1] Let the width of the *zhi ji* be a, and the length b. That is, $ab = 12$. (G)

[2] Let the width and the length of the *zhi ji* be the *gou* and the *gu* of a right triangle. Then, the diagonal of the *zhi ji* is the *xian*. That is, $a + c = 8$. (G)

[3] The expression in modern form is the equation: $x^3 + 4x - 28x + 23 = 0$. (C)

2. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the *xian* less the *gou*.

Ans. 2 *bu*.

Process. Let the element *tian* be the *xian* less the *gou*. From the statement we have 72 for the negative *shi*, 64 for the positive *fang*, 16 for the nega-

【注释】

[1] 开方式的现代形式为：$x^3-16x^2+64x-72=0$。（陈）

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：勾弦较为多少？

答：2步。

术：设天元一为勾弦较，以如积方法求其解。得到−72作为常数项，64作为一次项系数，−16作为二次项系数，1作为最高次项系数，开立方。符合所问。

3.

【原文】

今有直积一十二步。只云勾弦和八步。问：股弦较几何？

答曰：一步。

术曰：立天元一为股弦较，如积求之，得九为正实，二十八为益方，二十为从廉，一为益隅，立方开之。[1] 合问。

【注释】

[1] 开方式的现代形式为：$-x^3+20x^2-28x+9=0$。（陈）

【今译】

今有长方形的面积12步。只知道勾弦和为8步。问：股弦较为多少？

答：1步。

术：设天元一为股弦较，以如积方法求其解。得到9作为常数项，−28作为一次项系数，20作为二次项系数，−1作为最高次项系数，开立方。符合所问。

tive *lian*, and 1 for the positive *yu*, a cubic expression[1] whose root is the required number.

[Notes]

[1] The expression in modern form is the equation: $x^3 - 16x^2 + 64x - 72 = 0$.

(C)

3. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the *xian* less the *gu*.

Ans. 1 *bu*.

Process. Let the element *tian* be the *xian* less the *gu*. From the statement we have 9 for the positive *shi*, 28 for the negative *fang*, 20 for the positive *lian*, and 1 for the negative *yu*, a cubic expression[1] whose root is the required number.

[Notes]

[1] The expression in modern form is the equation: $-x^3 + 20x^2 - 28x + 9 = 0$.

(C)

4.

【原文】

今有直积一十二步。只云勾弦和八步。问：弦和较[1]几何？

答曰：二步。

术曰：立天元一为弦和较，如积求之，得七十二为正实，七十二为益方，一十六为从廉，一为正隅，立方开之。[2]合问。

【注释】

[1] 弦和较即弦与勾股和之差：$(a+b)-c$。（郭）

[2] 开方式的现代形式为：$x^3+16x^2-72x+72=0$。（陈）

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：弦和较为多少？

答：2步。

术：设天元一为弦和较，以如积方法求其解。得到72作为常数项，−72作为一次项系数，16作为二次项系数，1作为最高次项系数，开立方。符合所问。

5.

【原文】

今有直积一十二步。只云勾弦和八步。问：弦较较[1]几何？

答曰：四步。

术曰：立天元一为弦较较，如积求之，得一百九十二为益实，一百二十八为从方，二十四为益廉，一为正隅，立方开之。[2]合问。

【注释】

[1] 弦较较即弦与勾股差之差：$c-(b-a)$。（郭）

4. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the *gu* plus the *gou* minus the *xian* [1].

Ans. 2 *bu*.

Process. Let the element *tian* be the sum of the *gu* and the *gou* less the *xian*. From the statement we have 72 for the positive *shi*, 72 for the negative *fang*, 16 for the positive *lian*, and 1 for the positive *yu*, a cubic expression [2] whose root is the required number.

[Notes]

[1] That is, $(a+b)-c$. (C)

[2] The expression in modern form is the equation: $x^3+16x^2-72x+72=0$. (C)

5. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the *xian* less the *jiao* [1].

Ans. 4 *bu*.

Process. Let the element *tian* be the *xian* less the *jiao*. From the statement we have 192 for the negative *shi*, 128 for the positive *fang*, 24 for the negative *lian*, and 1 for the positive *yu*, a cubic expression [2] whose root is the required number.

[2] 开方式的现代形式为：$x^3-24x^2+128x-192=0$。(陈)

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：弦较较为多少？

答：4步。

术：设天元一为弦较较，以如积方法求其解。得到 −192 作为常数项，128作为一次项系数，−24作为二次项系数，1作为最高次项系数，开立方。符合所问。

6.

【原文】

今有直积一十二步。只云勾弦和八步。问：勾较较[1]几何？

答曰：二步。

术曰：立天元一为勾较较，如积求之，得八为正实，八为从方，八为益廉，一为正隅，立方开之。[2]合问。

【注释】

[1] 勾较较即勾与勾股差之差 $a-(b-a)$。(郭)

[2] 开方式的现代形式为：$x^3-8x^2+8x+8=0$。(陈)

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：勾较较为多少？

答：4步。

术：设天元一为勾较较，以如积方法求其解。得到8作为常数项，8作为一次项系数，−8作为二次项系数，1作为最高次项系数，开立方。符合所问。

【Notes】

[1] The *xian* less the *jiao* is the difference between the *xian* and the difference of the *gou* and the *gu*. That is, $c - (b - a)$. (G)

[2] The expression in modern form is the equation: $x^3 - 24x^2 + 128x - 192 = 0$. (C)

6. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the *gou* less the *jiao* [1].

Ans. 2 *bu*.

Process. Let the element *tian* be the *gou* less the *jiao*. From the statement we have 8 for the positive *shi*, 8 for the positive *fang*, 8 for the negative *lian*, and 1 for the positive *yu*, a cubic expression [2] whose root is the required number.

【Notes】

[1] The *gou* less the *jiao* is the difference between the *gou* and the difference of the *gou* and the *gu*. That is, $a - (b - a)$. (G)

[2] The expression in modern form is the equation: $x^3 - 8x^2 + 8x + 8 = 0$. (C)

7.

【原文】

今有直积一十二步。只云勾弦和八步。问：勾股和几何？

答曰：七步。

术曰：立天元一为勾股和，如积求之，得五百五十三为正实，一百为益方，四为益廉，一为正隅，立方开之。[1] 合问。

【注释】

[1] 开方式的现代形式为：$x^3 - 4x^2 - 100x + 553 = 0$。（陈）

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：勾股和为多少？

答：7步。

术：设天元一为勾股和，以如积方法求其解。得到553作为常数项，-100作为一次项系数，-4作为二次项系数，1作为最高次项系数，开立方。符合所问。

8.

【原文】

今有直积一十二步。只云勾弦和八步。问：股弦和几何？

答曰：九步。

术曰：立天元一为股弦和，如积求之，得九为正实，一百为益方，二十为从廉，一为益隅，立方开之。[1] 合问。

【注释】

[1] 开方式的现代形式为：$-x^3 + 20x^2 - 100x + 9 = 0$。（陈）

7. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the sum of the *gou* and the *gu*.

Ans. 7 *bu*.

Process. Let the element *tian* be the sum of the *gou* and the *gu*. From the statement we have 553 for the positive *shi*, 100 for the negative *fang*, 4 for the negative *lian*, and 1 for the positive *yu*, a cubic expression [1] whose root is the required number.

[Notes]

[1] The expression in modern form is the equation: $x^3 - 4x^2 - 100x + 553 = 0$. (C)

8. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the sum of the *gu* and the *xian*.

Ans, 9 *bu*.

Process. Let the element *tian* be the sum of the *gu* and the *xian*. From the statement we have 9 for the positive *shi*, 100 for the negative *fang*, 20 for the positive *lian*, and 1 for the negative *yu*, a cubic expression [1] whose root is the required number.

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：股弦和为多少？

答：9步。

术：设天元一为股弦和，以如积方法求其解。得到9作为常数项，-100作为一次项系数，20作为二次项系数，-1作为最高次项系数，开立方。符合所问。

9.

【原文】

今有直积一十二步。只云勾弦和八步。问：弦较和[1]几何？

答曰：六步。

术曰：立天元一为弦较和，如积求之，得七十二为正实，七十二为益方，一十六为从廉，一为益隅，立方开之。[2]合问。

【注释】

[1] 弦较和即弦与勾股差之和：$(b-a)+c$。（郭）

[2] 开方式的现代形式为：$-x^3+16x^2-72x+72=0$。（陈）

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：弦较和为多少？

答：6步。

术：设天元一为弦较和，以如积方法求其解。得到72作为常数项，-72作为一次项系数，16作为二次项系数，-1作为最高次项系数，开立方。符合所问。

【Notes】

[1] The expression in modern form is the equation: $-x^3 + 20x^2 - 100x + 9 = 0$. (C)

9. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the sum of the *xian* and the *jiao*[1].

Ans. 6 *bu*.

Process. Let the element *tian* be the sum of the *xian* and the *jiao*. From the statement we have 72 for the positive *shi*, 72 for the negative *fang*, 16 for the positive *lian*, and 1 for the negative *yu*, a cubic expression[2] whose root is the required number.

【Notes】

[1] The sum of the *xian* and the *jiao* is the sum of the *xian* and the difference between the *gou* and the *gu*. That is, $(b - a) + c$. (G)

[2] The expression in modern form is the equation: $-x^3 + 16x^2 - 72x + 72 = 0$. (C)

10.

【原文】

今有直积一十二步。只云勾弦和八步。问：弦和和[1]几何？

答曰：一十二步。

术曰：立天元一为弦和和，如积求之，得一百九十二为益实，一百二十八为益方，二十四为从廉，一为益隅，立方开之。[2]合问。

【注释】

[1] 弦和和即弦与勾股和之和：$c+(a+b)$。（郭）

[2] 开方式的现代形式为：$-x^3+24x^2-128x-192=0$。（陈）

【今译】

今有长方形的面积12步。只云勾弦和为8步。问：弦和和为多少？

答：12步。

术：设天元一为弦和和，以如积方法求其解。得到−192作为常数项，−128作为一次项系数，24作为二次项系数，−1作为最高次项系数，开立方。符合所问。

11.

【原文】

今有直积一百二十步。只云黄方乘句股较，得四十二步。[1]问：勾及黄方各几何？

答曰：勾八步，黄方六步。

术曰：立天元一为勾，如积求之，得二万八百为正实，五百一十七为益上廉，三为从隅，三乘方开之，[2]得勾。又：立天元一为黄方，如积求之，得五万五百四十四为正实，一千四百四十为益上廉，一为从隅，三乘方开之。[3]合问。

【注释】

[1] 黄方即$(a+b)-c$。此即：$ab=120$；$(a+b-c)(b-a)=42$。（郭）

[2] 开方式的现代形式为：$3x^4-517x^2+20800=0$。（陈）

[3] 开方式的现代形式为：$x^4-1440x^2+50544=0$。（陈）

10. The *zhi ji* is 12 (square) *bu*, and the sum of the *gou* and the *xian* 8 *bu*. Find the sum of the *xian* and the *he* [1].

Ans. 12 *bu*.

Process. Let the element *tian* be the sum of the *xian* and the *he*. From the statement we have 192 for the negative *shi*, 128 for the negative *fang*, 24 for the positive *lian*, and 1 for the negative *yu*, a cubic expression[2] whose root is the required number.

【Notes】

[1] The sum of the *xian* and the *he* is the sum of the *xian* and the sum of the *gou* and the *gu*. That is, $c + (a + b)$. (G)

[2] The expression in modern form is the equation: $-x^3 + 24x^2 - 128x - 192=0$. (C)

11. The *zhi ji* is 120 (square) *bu*, and the product of the *huang fang* by the *jiao* 42 *bu*. [1] Find the *gou* and the *huang fang*.

Ans. *Gou*, 8 *bu*; *huang fang*, 6 *bu*.

Process. Let the element *tian* be the *gou*. From the statement we have 20800 for the positive *shi*, 517 for the negative upper *lian*, and 3 for the positive *yu*, an expression [2] of the fourth degree whose root is the required *gou*. Again let the element *tian* be the *huang fang*. From the statement we have 50544 for the positive *shi*, 1440 for the negative upper *lian*, and 1 for the positive *yu*, an expression [3] of the fourth degree whose root is the required *huang fang*.

【今译】

今有长方形的面积120步。只云黄方与勾股较之积等于42步。问：勾与黄方各为多少？

答：勾8步，黄方6步。

术：设天元一为勾，以如积方法求其解。得到20800作为常数项，−517作为二次项系数，3作为最高次项系数，开四次方，得到勾。又：设天元一为黄方，以如积方法求其解。得到50544作为常数项，−1440作为二次项系数，1作为最高次项系数，开四次方，便得到黄方。符合所问。

12.

【原文】

今有直积一百二十步。只云弦较较乘弦和较，得六十步。[1] 问：黄方几何？

答曰：六步。

术曰：立天元一为黄方，如积求之，得三十六为益实，一为从隅，平方开之。[2] 合问。

【注释】

[1] 此即：$[c-(b-a)][(a+b)-c]=60$。（郭）

[2] 开方式的现代形式为：$x^2-36=0$。（陈）

【今译】

今有长方形的面积120步。只云弦较较与弦和较之积等于60步。问：黄方为多少？

答：6步。

术：设天元一为黄方，以如积方法求其解。得到−36作为常数项，1作为二次项系数，开平方。符合所问。

[Notes]

[1] That is, $ab = 120$; the *huang fang* is $(a+b)-c$, $(a+b-c)(b-a) = 42$. (G)

[2] The expression in modern form is the equation: $3x^4 - 517x^2 + 20800 = 0$. (C)

[3] The expression in modern form is the equation: $x^4 - 1440x^2 + 50544 = 0$. (C)

12. The *zhi ji* is 120 (square) *bu*, and the product of the *xian* less the *jiao* by the *he* less the *xian* 60 *bu*. [1] Find the *huang fang*.

Ans, 6 *bu*.

Process. Let the element *tian* be the *huang fang*. From the statement we have 36 for the negative *shi*, and 1 for the positive *yu*, a quadratic expression [2] whose root is the required *huang fang*.

[Notes]

[1] That is, $[c-(b-a)][(a+b)-c] = 60$. (G)

[2] The expression in modern form is the equation: $x^2 - 36 = 0$. (C)

13.

【原文】

今有直积一百六十八步。只云勾弦较乘弦和较，得一百八步。[1] 问：勾弦较几何？

答曰：一十八步。

术曰：立天元一为勾弦较，如积求之，得一万四百九十七步六分为益实，二百九十一步六分为益上廉，一为正隅，三乘方开之。[2] 合问。

【注释】

[1] 此即：$ab = 168$，$(c-a)[(a+b)-c] = 108$。（郭）

[2]“步”下“六分”，即0.6步。（郭）开方式的现代形式为：

$x^4 - 291.6x^2 - 10497.6 = 0$。（陈）

【今译】

今有长方形的面积168步。只云勾弦较与弦和较之积等于108步。问：勾弦较为多少？

答：勾8步，黄方6步。

术：设天元一为勾弦较，以如积方法求其解。得到 $-10497\frac{6}{10}$ 作为常数项，$-291\frac{6}{10}$ 作为二次项系数，1作为最高次项系数，开四次方，便得到勾弦较。符合所问。

14.

【原文】

今有直积一百八步。只云勾弦较乘勾股较，得一十八步。[1] 问：勾几何？

答曰：九步。

术曰：立天元一为勾，如积求之，得三百七十七万九千一百三十六为

13. The *zhi ji* is 168 (square) *bu*, and the product of the *xian* less the *gou* by the *he* less the *xian* 108 *bu*.[1] Find the *xian* less the *gou*.

Ans, 18 *bu*.

Process. Let the element *tian* be the *xian* less the *gou*. From the statement we have 10497 *bu* 6 *fen* for the negative *shi*, 291 *bu* 6 *fen* for the negative upper *lian*, and 1 for the positive *yu*, an expression[2] of the fourth degree whose root is the required number.

【 Notes 】

[1] That is, $ab = 168$, $(c - a)[(a + b) - c] = 108$. (G)

[2] 6 *fen* following *bu* is 0.6 *bu*. (G)

The expression in modern form is the equation: $x^4 - 291.6x^2 - 10497.6 = 0$. (C)

14. The *zhi ji* is 108 (square) *bu*, and the product of the *xian* less the *gou* by the *gu* less the *gou* 18 *bu*. [1] Find the *gou*.

Ans, 9 *bu*.

Process. Let the element *tian* be the *gou*. From the statement we have 3779136 for the negative *shi*, 69984 for the positive upper *lian*, 207 for the

益实，六万九千九百八十四为从上廉，二百七为益三廉，一为益隅，五乘方开之。[2] 合问。

【注释】

[1] 此即：$ab = 168$，$(c-a)(b-a) = 18$。(郭)

[2] 开方式的现代形式为：$-x^6 - 207x^4 + 69984x^2 - 3799136 = 0$。(陈)

【今译】

今有长方形的面积108步。只云勾弦较与勾股较之积等于18步。问：勾为多少?

答：9步。

术：设天元一为勾，以如积方法求其解。得到−3799136作为常数项，69984作为二次项系数，−207作为四次项系数，−1作为最高次项系数，开六次方，便得到勾。符合所问。

15.

【原文】

今有直积一百六十八步。只云股弦较乘勾弦较，得一十八步。[1] 问：弦和和几何?

答曰：五十六步。

术曰：立天元一为弦和和，如积求之，得三千一百三十六为益实，一为从隅，平方开之。[2] 合问。

【注释】

[1] 此即：$(c-b)(c-a) = 18$。(郭)

[2] 开方式的现代形式为：$x^2 - 3136 = 0$。(陈)

【今译】

今有长方形的面积168步。只云股弦较与勾弦较之积等于18步。问：弦和和为多少?

答：56步。

术：设天元一为弦和和，以如积方法求其解。得到−3136作为常数项，1作为最高次项系数，开平方，便得到弦和和。符合所问。

negative third *lian*, and 1 for the negative *yu*, an expression[2] of the sixth degree whose root is the required number.

[Notes]

[1] That is, $ab = 168$, $(c - a)(b - a) = 18$. (G)

[2] The expression in modern form is the equation: $-x^6 - 207x^4 + 69984x^2 - 3799136 = 0$. (C)

15. The *zhi ji* is 168 (square) *bu*, and the product of the *xian* less the *gu* by the *xian* less the *gou* 18 *bu*. [1] Find the sum of the *xian* and the *he*.

Ans. 56 *bu*.

Process. Let the element *tian* be the sum of the *xian* and the *he*. From the statement we have 3136 for the negative *shi*, and 1 for the positive *yu*, a quadratic expression [2] whose root is the required number.

[Notes]

[1] That is, $(c - b)(c - a) = 18$. (G)

[2] The expression in modern form is the equation: $x^2 - 3136 = 0$. (C)

16.

【原文】

今有直积一百八步。只云弦较较乘勾弦较，得七十二步。[1] 问：弦和较几何？

答曰：六步。

术曰：立天元一为弦和较，如积求之，得五千一百八十四为益实，一百四十四为从隅，平方开之。[2] 合问。

【注释】

[1] 此即：$[c-(b-a)](c-a)=72$。（郭）

[2] 开方式的现代形式为：$144x^2-5184=0$。（陈）

【今译】

今有长方形的面积108步。只云弦较较与勾弦较之积等于72步。问：弦和较为多少？

答：6步。

术：设天元一为弦和较，以如积方法求其解。得到−5184作为常数项，144作为最高次项系数，开平方，便得到弦和和。符合所问。

17.

【原文】

今有直积一百二十步。只云股弦和乘勾弦较，得二百八十八步。[1] 问：黄方几何？

答曰：六步。

术曰：立天元一为半黄方，如积求之，得三千六百为正实，四百九为益上廉，一为从隅，三乘方开之，[2] 得半黄方面。倍之。合问。

16. The *zhi ji* is 108 (square) *bu*, and the product of the *xian* less the *jiao* by the *xian* less the *gou* 72 *bu*. [1] Find the *he* less the *xian*.

Ans. 6 *bu*.

Process. Let the element *tian* be the *he* less the *xian*. From the statement we have 5184 for the negative *shi*, and 144 for the positive *yu*, a quadratic expression [2] whose root is the required number.

【 Notes 】

[1] That is, $[c-(b-a)](c-a)=72$. (G)

[2] The expression in modern form is the equation: $144x^2-5184=0$. (C)

17. The *zhi ji* is 120 (square) *bu*, and the product of the sum of the *xian* and the *gu* by the *xian* less the *gou* 288 *bu*. [1] Find the *huang fang*.

Ans. 6 *bu*.

Process. Let the element *tian* be the half of the *huang fang*. From the statement we have 3600 for the positive *shi*, 409 for the negative upper *lian*, and 1 for the positive *yu*, an expression [2] of fourth degree whose root is the half of the *huang fang*. The root multiplied by 2 gives the *huang fang*.

【注释】

[1] 此即：$(c+b)(c-a)=288$。（郭）

[2] 开方式的现代形式为：$x^4-409x^2+3600=0$。（陈）

【今译】

今有长方形的面积120步。只云股弦和与勾弦较之积等于288步。问：黄方为多少？

答：6步。

术：设天元一为黄方的$\frac{1}{2}$，以如积方法求其解。得到3600作为常数项，−409作为二次项系数，1作为最高次项系数，开四次方，得到黄方的$\frac{1}{2}$。加倍，便得到黄方。符合所问。

18.

【原文】

今有直积一百二十步。只云勾弦和乘股弦较，得五十步。[1]问：黄方及勾股和几何？

答曰：黄方六步，勾股和二十三步。

术曰：立天元一为黄方，如积求之，得五万七千六百为正实，一千六百三十六为益上廉，一为正隅，三乘方开之，[2]得黄方。又：立天元一为勾股和，如积求之，得一千八百四十为正实，八十为益方，开无隅平方除之。[3]合问。

【注释】

[1] 此即：$(c+a)(c-b)=50$。（郭）

[2] 开方式的现代形式为：$x^4-1636x^2+57600=0$。（陈）

[3] 开方式的现代形式为：$-80x+1840=0$。（陈）

[Notes]

[1] That is, $(c+b)(c-a)=288$. (G)

[2] The expression in modern form is the equation: $x^4-409x^2+3600=0$. (C)

18. The *zhi ji* 120 (square) *bu*, and the product of the sum of the *xian* and the *gou* by the *xian* less the *gu* 50 *bu*.[1] Find the *huang fang* and the sum of the *gou* and the *gu*.

Ans. *Huang fang*, 6 *bu*;

sum of the *gou* and the *gu*, 23 *bu*.

Process. Let the element *tian* be the *huang fang*. From the statement we have 57600 for the positive *shi*, 1636 for the negative upper *lian*, and 1 for the positive *yu*, an expression[2] of fourth degree whose root is the *huang fang*. Again let the element *tian* be the sum of the *gou* and the *gu*. From the statement we have 1840 for the positive *shi*, and 80 for the negative *fang*, a linear expression[3] whose root is the sum.

【今译】

今有长方形的面积120步。只云勾弦和与股弦较之积等于50步。问：黄方及勾股和为多少？

答：黄方6步，勾股和23步。

术：设天元一为黄方，以如积方法求其解。得到57600作为常数项，−1636作为二次项系数，1作为最高次项系数，开四次方，得到黄方。

又：设天元一为勾股和，以如积方法求其解。得到1840作为常数项，−80作为一次项系数，开无隅平方。符合所问。

19.

【原文】

今有直积一百八步。只云勾弦和乘股弦和，得六百四十八步。[1] 问：黄方及弦各几何？

答曰：黄方六步，弦一十五步。

术曰：立天元一为黄方，如积求之，得二万三千三百二十八为益实，六百四十八为从隅，平方开之，[2] 得黄方六步。　又：立天元一为弦，如积求之，得一千八十为正实，七十二为益方，上实，下法而一，得弦。[3] 合问。

【注释】

[1] 此即：$(c+a)(c+b)=648$。（郭）

[2] 开方式的现代形式为：$648x^2-23328=0$。（陈）

[3] 开方式的现代形式为：$-72x+1080=0$。（陈）

【今译】

今有长方形的面积108步。只云勾弦和与股弦和之积等于648步。问：黄方及弦各为多少？

答：黄方6步，弦15步。

术：设天元一为黄方，以如积方法求其解。得到−23328作为常数项，648作为最高次项系数，开平方，得到黄方6步。　又：设天元一为弦，以如积方法求其解。得到1080作为常数项，−72作为一次项系数，常数项作为被除数，一次项系数作为除数，除之，便得到弦。符合所问。

[Notes]

[1] That is, $(c + a)(c - b) = 50$. (G)

[2] The expression in modern form is the equation: $x^4 - 1636x^2 + 57600 = 0$. (C)

[3] The expression in modern form is the equation: $-80x + 1840 = 0$. (C)

19. The *zhi ji* is 108 (square) *bu*, and the product of the sum of the *xian* and the *gou* by the sum of the *xian* and the *gu* 648 *bu*.[1] Find the *huang fang* and the *xian*.

Ans. *Huang fang*, 6 *bu*;

xian, 15 *bu*.

Process. Let the element *tian* be the *huang fang*. From the statement we have 23328 for the negative *shi*, and 648 for the positive *yu*, a quadratic expression[2] whose root, 6 *bu*, is the *huang fang*. Again let the element *tian* be the *xian*. From the statement we have 1080 for the positive *shi*, and 72 for the negative *fang*, a linear expression[3] whose root is the *xian*.

[Notes]

[1] That is, $(c + a)(c + b) = 648$. (G)

[2] The expression in modern form is the equation: $648x^2 - 23328 = 0$. (C)

[3] The expression in modern form is the equation: $-72x + 1080 = 0$. (C)

20.

【原文】

今有直积一百八步。只云五和、五较相乘，得三千七百八十步。[1] 问：弦几何？

答曰：一十五步。

术曰：立天元一为弦，如积求之，得七千九百七十四亿九千三百六十五万六百二十五为益实，七十五亿二千二百八十四万二千六百为从上廉，二千五十二万四千九百九十五为益三廉，一万二千六百三十六为从隅，五乘方开之，[2] 得弦。合问。

【注释】

[1] 此即：

$$\{(a+b)+(c+a)+(c+b)+[c+(a+b)]+[c+(b-a)]\}\times\{(b-a)+(c-a)+(c-b)+[c-(b-a)]+[(a+b)-c]\}=3780。$$

（郭）

[2] 开方式的现代形式为：

$12636x^6-20524995x^4+7522842600x^2-797493650625=0$。（陈）

【今译】

今有长方形的面积108步。只云五和与五较之积等于3780步。问：弦为多少？

答：15步。

术：设天元一为弦，以如积方法求其解。得到−797493650625作为常数项，7522842600作为二次项系数，−20524995作为四次项系数，12636作为最高次项系数，开六次方，便得到弦。符合所问。

20. The *zhi ji* is 108 (square) *bu*, and the product of the five *he* by the five *jiao* 3780 *bu*. [1] Find the *xian*.

Ans. 15 *bu*.

Process. Let the element *tian* be the *xian*. From the statement we have 797493650625 for the negative *shi*, 7522842600 for the positive upper *lian*, 20524995 for the negative third *lian*, and 12636 for the positive *yu*, an expression [2] of sixth degree whose root is the required number.

[Notes]

[1] That is, $\{(a+b)+(c+a)+(c+b)+[c+(a+b)]+[c+(b-a)]\}\times\{(b-a)+(c-a)+(c-b)+[c-(b-a)]+[(a+b)-c]\}=3780$. (G)

[2] The expression in modern form is the equation: $12636x^6-20524995x^4+7522842600x^2-797493650625=0$. (C)